ON THE ESTIMATION OF POINCARÉ MAPS OF THREE-DIMENSIONAL VECTOR FIELDS NEAR A HYPERBOLIC CRITICAL POINT

By

Yuting Zou

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ABSTRACT

ON THE ESTIMATION OF POINCARÉ MAPS OF THREE-DIMENSIONAL VECTOR FIELDS NEAR A HYPERBOLIC CRITICAL POINT

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We study the estimation of Poincaré maps of three-dimensional vector fields near a hyperbolic critical point, which involves linearization problems. Standard linearization theorems have several defects in applications. They usually require complicated non-resonance conditions on the eigenvalues of the vector field at the critical point. Even when one has these non-resonance conditions, as one gets close to a resonance, the size of the neighborhood where the C^1 linearization exists typically gets too small for practical uses. We seek for a linearization theorem that overcomes these shortcomings and may have broad practical applications.

We have proved a partial linearization theorem that gives a C^1 linearization h near a hyperbolic critical point p on a two-dimensional invariant surface Σ of a three-dimensional vector field X. Let the eigenvalues of $DX(p)$ be a, b and c, where $a > 0 > b > c$. Essentially our theorem only requires that $2b > c$ to obtain h in some neighborhood U of p in Σ. In addition, the explicit size of U is found, which depends on the C^2 information of X, as well as the C^0 and C^1 sizes of h. Based on our partial linearization theorem, we obtain desired estimation of Poincaré maps from some transversal curve to the stable manifold of p to another transversal surface to the unstable manifold of p.

Our estimation of such Poincaré maps will have many applications, including an

in-depth study of the famous Lorenz equations. For example, it seems likely that we will be able to substantially improve results of Tucker on the existence of the Lorenz strange attractor, and obtain rigorous results on the existence of chaos near the first homoclinic bifurcation as numerically investigated in the well-known book of Colin Sparrow.

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Introduction

Let M be a smooth (C^∞) three dimensional Riemannian manifold with associated topological metric d, and let X be a C^2 vector field on M with a hyperbolic critical point $p_0 \in M$ with real distinct eigenvalues a, b, c satisfying

$$a > 0 > b > c. \tag{1.1}$$

Assume that X is forward complete in the sense that, if $\phi(t, x)$ denotes the local flow of X, then, $\phi(t, x)$ is defined for all $t \geq 0$ and all $x \in M$.

Let $W^s(p_0), W^{ss}(p_0), W^u(p_0)$ denote, respectively, the stable, strong stable, and unstable manifolds of p_0.

These are defined as follows.

$$W^s(p_0) = \{x \in M : \phi(t, x) \text{ exists for all } t \geq 0 \text{ and } \lim_{t\to\infty} d(\phi(t, x), p_0) = 0\}$$

$$\begin{aligned} W^{ss}(p_0) &= \{x \in M : \phi(t,x) \text{ exists for all } t \geq 0 \\ &\qquad \text{and } \limsup_{t\to\infty} \frac{1}{t} \log d(\phi(t,x), p_0) < b\} \end{aligned}$$

$$W^u(p_0) = \{x \in M : \phi(t,x) \text{ exists for all } t \leq 0 \text{ and } \lim_{t\to-\infty} d(\phi(t,x), p_0) = 0\}.$$

It is well-known that $W^s(p_0)$, $W^{ss}(p_0)$ and $W^u(p_0)$ are C^2 injectively immersed copies of the plane and line, respectively, in M.

Let us use the notation $Comp(E, x)$ for the connected component of the set E containing the point x. For a point $p \in W^s(p_0)$ and a neighborhood U of p in M, let $W^s(p, U) = Comp(W^s(p_0) \bigcap U, p)$. One can choose arbitrarily small neighborhoods U so that $W^s(p, U)$ is a topological 2-disk and $U \setminus W^s(p, U)$ consists of a disjoint pair of open 3-balls.

Let $p \in W^s(p_0) \setminus \{p_0\}$, $q \in W^u(p_0) \setminus \{p_0\}$ and let N be a smoothly embedded 2-disk in M, such that N is transversal to $W^u(p_0)$ at q. An application of the Grobman-Hartman theorem (C^0 local linearization theorem) shows that there is a small neighborhood U of p in M and a connected component U_0 of $U \setminus W^s(p, U)$ such that there is a well-defined Poincaré map P from U_0 into N. This map simply takes a point in U_0 to the first point on its positive orbit which hits N.

In many applications (e.g. Section 2.2 in [7], [9], and the books [6], [1]) one encounters the following problem.

Let $p \in W^s(p_0) \setminus W^{ss}(p_0)$ and let γ be a C^2 embedded arc in U meeting

$W^s(p_0)$ transversely at p and assume that U is small enough so that for each $y \in \gamma$, $d(y, p)$ is boundedly related to the arc length of the subarc of γ joining p to y. Let $\gamma_0 = \gamma \bigcap U_0$, and, for a point $y \in \gamma_0$, let $\gamma_0'(y)$ denote the unit tangent vector to γ_0 at y. One seeks an estimate of the derivative $P'(y) = DP_y(\gamma_0'(y))$.

If the vector field X is C^1 linearizable near p_0, then it is well-known and easy to prove that one can find constants $0 < C_1 < C_2$ such that

$$C_1|P'(y)| \leq d(y,p)^{|\frac{b}{a}|-1} \leq C_2|P'(y)|. \tag{1.2}$$

In most applications using estimates of the type (1.2), one merely assumes that X is, in fact, at least C^1 linearizable near p_0. In view of the Sternberg linearization theorem, a generic C^∞ vector field is C^∞ linearizable near a hyperbolic critical point, so it does not seem to be a big assumption to assume linearity. However, it should be noted that there are several defects to the application of the Sternberg theorem even to get C^1 linearizations. Among these defects are the following.

- The theorem requires so-called non-resonance conditions on the eigenvalues a, b, c;
- Even in the case that X is C^∞, as one gets close to a resonance, the size of the neighborhood on which the C^1 linearization exists typically gets very small;
- The amount of smoothness required on X depends on the eigenvalues a, b, c.

Our main result shows that, with the mild additional assumption that $b > \frac{c}{2}$, we obtain the estimate (1.2) above for general C^2 vector fields in dimension three.

Theorem 1.0.1. *Let X, p_0, p, γ_0 and P be as above. Also, as above, for $y \in \gamma_0$, let*

$$P'(y) = DP_y(\gamma_0'(y))$$

be the derivative of P on the unit tangent vector $\gamma_0'(y)$ to γ_0 at y.

Assume that the eigenvalues a, b, c of X at the critical point p_0 satisfy

$$a > 0 > 2b > c. \tag{1.3}$$

Then, there are constants $0 < C_1 < C_2$ such that

$$C_1|P'(y)| \leq d(y,p)^{|\frac{b}{a}|-1} \leq C_2|P'(y)|. \tag{1.4}$$

The main tools we use are modifications of techniques in so-called *normal hyperbolicity theory* to get a C^2 invariant surface Σ which contains both γ_0 and p_0 and is tangent at p_0 to the sum of the eigenspaces of a, b. Since, by a theorem of P. Hartman [2], two dimensional C^2 vector fields with a hyperbolic saddle point p_0 are C^1 linearizable near p_0, one can get the required estimate.

Since we obtain a linearization of X restricted to an invariant two-dimensional surface through p_0 and not in a full neighborhood of p_0, we will refer to our result as a *partial linearization theorem.*

Remarks.

- It should be noted that the general tools of normal hyperbolicity in the literature give *some* two dimensional surfaces tangent at p_0 to the sum of the eigenspaces of a, b. However, these surfaces are far from unique, and are usually constructed by means of globalization techniques. Since we need such surfaces which contain an a priori given curve γ_0, the globalization technique is not applicable as far as we know. The existence of C^2 invariant surfaces through p_0 which contain an arbitrary given transverse curve γ_0 seems to be a new result, and our proof avoids the use of globalizations.

- For various applications, it is important to know the sizes of the constants C_1, C_2 in (1.2). We take care to give explicit estimates for these in our work below.

Using appropriate local coordinates near p_0 and replacing p and q by points in their orbits, it suffices to prove our results for vector fields in R^3 with a hyperbolic critical point at the origin. The precise statement needed is in Theorem 3.0.6.

We first review some of the history of linearization theorems.

Chapter 2

Linearization theorems

The first general linearization theorem was given by Poincaré in his thesis. He obtained a theorem that can be rephrased as follows:

Theorem 2.0.2. *Let*

$$\frac{dx_i}{dt} = \lambda_i x_i + f_i(x_1, \ldots, x_n), \quad 1 \leq i \leq n. \tag{2.1}$$

be a system of differential equations such that λ_i is a complex number, and each f_i is a complex analytic function in a neighborhood of the origin which vanishes together with its first order partial derivatives at the origin.

Suppose all the λ_i lie in the same open half-plane about the origin and

$$\lambda_i \neq \sum_{j=1}^{n} m_j \lambda_j \tag{2.2}$$

for any non-negative integers m_j with $2 \leq \sum_{j=1}^{k} m_j$.

Then, there exists a complex analytic change of coordinates $y_i = \psi_i(x_1, x_2, \cdots, x_n)$ *transforming (2.1) into the linear system*

$$dy_i/dt = \lambda_i y_i, \quad 1 \leq i \leq n$$

The conditions (2.2) are called *non-resonance conditions* and, without them, there are examples even of polynomials f_i where the theorem fails.

Let us recall the definitions of local linearizations in both the flow and diffeomorphism contexts.

Let $f : R^n \to R^n$ be a C^r diffeomorphism, $r \geq 1$, from a neighborhood U of 0 in R^n into R^n such that

$$f(0) = 0, \; Df(0) = L.$$

Then, L is, of course, a linear automorphism of R^n.

A *local* C^k *linearization* of f near 0 is a C^k homeomorphism h from a neighborhood V of 0 in U to another neighborhood V' of 0 such that

$$Lh(x) = hf(x)$$

for $x \in V \bigcap f^{-1}(V)$.

Next, consider a C^r local flow near 0 in R^n with a fixed point (or critical point) at 0. This is a C^r map $\phi(t, x)$ from a neighborhood $(-\epsilon, \epsilon) \times U$ into R^n where ϵ is a positive real number such that

1. $\phi(0, x) = x$ for all $x \in U$,

2. $\phi(t, 0) = 0$ for all $t \in (-\epsilon, \epsilon)$, and

3. $\phi(t + s, x) = \phi(t, \phi(s, x))$ for $t, s \in (-\epsilon, \epsilon)$ such that $t + s \in (-\epsilon, \epsilon)$.

We often use ϕ or ϕ_t to denote the local flow where $\phi_t(x) = \phi(t, x)$. Sometimes we reverse the t and x and write $\phi(x, t)$.

A C^r vector field X near 0 in R^n is a C^r map $X : U_0 \to R^n$ from a neighborhood U_0 of 0.

If X is a C^r vector field near 0 such that $X(0) = 0$ (with $r \leq 1$), then, for every $T > 0$ there is a neighborhood U of 0 in R^n and a local flow $\phi(t, x) = \phi_X(t, x)$ defined on $(-T, T) \times U$ such that

$$\frac{\partial \phi(t, x)}{\partial t} = X(\phi(t, x))$$

for all $t \in (-T, T)$ and $x \in U$.

The local flow $\phi(t, x)$ is called the *flow of X near 0*.

The local flow $\phi(t, x)$ is called *linear* if, in addition to the usual flow properties, we have $x \to \phi(t, x)$ is a linear automorphism of R^n for each t.

Let ϕ_t and ψ_t, $|t| < \epsilon$ be two local flows near 0 with fixed points at 0. Let $k \geq 0$ be a positive integer.

A *local C^k conjugacy near* 0 between ϕ_t and ψ_t is a C^k homeomorphism h from a neighborhood V of 0 to another neighborhood V' of 0 such that, there is a positive real number ϵ_1 such that

1. ψ_t is defined on V' for all $|t| \leq \epsilon_1$,

2. $h \circ \phi_t$ is defined for all $|t| \leq \epsilon_1$, and

3. $\psi_t \circ h = h \circ \phi_t$ for all $|t| \leq \epsilon_1$.

If ψ_t is linear, then a local C^k conjugacy between ϕ_t and ψ_t near 0 is called a *local linearization* of ϕ_t near 0.

If X is a C^1 local vector field near 0 with associated local flow ϕ_t and derivative $L = DX(0)$, then a local C^k linearization of X near 0 is a local C^k conjugacy between ϕ_t and the linear flow given by L. In this case, we may assume that the local flow ϕ_t is defined for $|t| \leq T$ where T is arbitrary. For large T, we simply have to shrink the neighborhood on which the linearization is defined.

We are now in a position to state the well-known Grobman-Hartman Theorem [6], [3].

Theorem 2.0.3. *(Grobman-Hartman).*

1. *Let f be a C^1 local diffeomorphism of R^n with a hyperbolic fixed point at 0 (i.e., the eigenvalues of $Df(0)$ have norm different from 1). Then, f has a local C^0 linearization near 0.*

2. *Let X be a C^1 local vector field near 0 with a hyperbolic critical point at 0 (i.e. $X(0) = 0$ and the eigenvalues of $DX(0)$ have non-zero real parts). Let $L = DX(0)$ and let ψ_t be the flow of L. Then, for any $T > 0$, there is a local C^0 conjugacy h between ϕ_t and ψ_t for all $|t| \leq T$.*

This is a beautiful and powerful theorem. One could ask for smooth local linearizations for smoother f and/or X.

In this direction there is another well-known theorem due to Sternberg. See the appendix starting on page 256 in [3] for a proof of this theorem as well as related results, and further references.

Theorem 2.0.4. *(Sternberg)*

1. *Let f be a C^∞ local diffeomorphism of R^n with a hyperbolic fixed point at 0 (i.e., the eigenvalues of $Df(0)$ have norm different from 1). Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $Df(0)$.*

   Suppose that, for every positive $1 \leq i \leq n$,and any n-tuple of non-negative integers $(m_1, m_2, \ldots, m_n)$ with $2 \leq \sum_{j=1}^{n} m_j$, we have

$$\lambda_i \neq \prod_{j=1}^{n} \lambda_j^{m_j}. \tag{2.3}$$

   Then, f has a local C^∞ linearization near 0.

2. *Let X be a C^∞ local vector field near 0 with a hyperbolic critical point at 0 (i.e. $X(0) = 0$ and the eigenvalues of $DX(0)$ have non-zero real parts). Let $L = DX(0)$ and let ψ_t be the flow of L.*

   Suppose that, for every positive $1 \leq i \leq n$,and any n-tuple of non-negative integers $(m_1, m_2, \ldots, m_n)$ with $2 \leq \sum_{j=1}^{n} m_j$, we have

$$\lambda_i \neq \sum_{j=1}^{n} m_j \lambda_j. \tag{2.4}$$

Then, for any $T > 0$, there is a local C^∞ conjugacy h between ϕ_t and ψ_t for all $|t| \leq T$.

Again the conditions (2.3) and (2.4) are called *non-resonance conditions*. Without them, even a real-analytic system may not be C^1 linearizable.

Sternberg's theorem is very important, and has many applications. However, as we mentioned in our introduction, it does have some shortcomings.

Finally, we mention the two dimensional C^1 linearization theorems of Hartman [2] which is of major relevance to this thesis. Here, fortunately, there are no non-resonance conditions to worry about.

Theorem 2.0.5. *1. Let f be a C^2 local diffeomorphism of R^2 with a hyperbolic saddle fixed point at 0 (i.e., a pair of real eigenvalues λ_1, λ_2 of $Df(0)$ satisfy $0 < |\lambda_2| < 1 < |\lambda_1|$). Then, f has a local C^1 linearization near 0.*

2. Let X be a C^2 local vector field near 0 in R^2 with a hyperbolic saddle critical point at 0 (i.e. $X(0) = 0$ and the eigenvalues λ_1, λ_2 of $L = DX(0)$ satisfy $\lambda_2 < 0 < \lambda_1$). Let ψ_t be the flow of L. Then, for any $T > 0$, there is a local C^1 conjugacy h between ϕ_t and ψ_t for all $|t| \leq T$.

As previously mentioned, we will give a new proof of Theorem 2.0.5 with explicit estimates for the C^0 and C^1 sizes of the linearization.

Chapter 3

A quantitative partial linearization theorem in R^3

Let $x = (x_1, x_2, x_3) \in R^3$, and $X(x)$ be a forward complete C^2 vector field which vanishes at the origin and can be written as

$$X(x) = (X_1(x), X_2(x), X_3(x)) = (ax_1 + \bar{X}_1(x), bx_2 + \bar{X}_2(x), cx_3 + \bar{X}_3(x))$$

where $c < 2b < 0 < a$, and $\bar{X}_{i,x_j}(0,0,0) = 0$ for $i, j = 1, 2, 3$.

For small positive ε_0, let

$$B_1 = B_{1\varepsilon_0} = \{x \in R^3 || x_2| \le \varepsilon_0, |x_1| \le \varepsilon_0, |x_3| \le \varepsilon_0\}$$

and let $|\cdot| = \sup_{x \in B_1} |\cdot|$ denote the maximum norm of various quantities in B_1. Then, there is a finite constant $M > 0$, such that $|\bar{X}_{i,x_j,x_k}(x)| \le M$ for

$i, j, k = 1, 2, 3$, and, using the mean value theorem, we have some small constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that $|\bar{X}_{i,x_j}(x)| \leq M \cdot \varepsilon_0 = \epsilon_2$, and $|\bar{X}_i(x)| \leq \epsilon_2 \cdot \varepsilon_0 = \epsilon_1$ for $i, j = 1, 2, 3$.

Let

$$B_{1top} = \{x \in B_1 | x_2 = \varepsilon_0\},$$

$$B_{1bot} = \{x \in B_1 | x_2 = -\varepsilon_0\},$$

$$B_1^+ = \{x \in B_1 | x_1 = \varepsilon_0\},$$

and

$$B_1^- = \{x \in B_1 | x_1 = -\varepsilon_0\}.$$

and let $\varphi(x,t) = \varphi_t(x)$ be the local flow of X, so we may write

$$\varphi(x,t) = (\varphi_1, \varphi_2, \varphi_3) = (e^{at}x_1 + \bar{\varphi}_1, e^{bt}x_2 + \bar{\varphi}_2, e^{ct}x_3 + \bar{\varphi}_3)$$

where $\bar{\varphi}_i$ for $i = 1, 2, 3$ are the higher order terms. Set $\psi_t(x)$ to be the flow of the corresponding linear vector field of X, i.e,

$$\psi_t(x) = (e^{at}x_1, e^{bt}x_2, e^{ct}x_3)$$

Let us denote the origin $(0,0,0)$ by $\mathbf{0}$, and let $W^s(\mathbf{0})$ denote its stable manifold.

Let $W^s_{loc} = W^s_{loc}(\mathbf{0}) = Comp(W^s(\mathbf{0}) \bigcap B_1, \mathbf{0})$ denote the connected component of the $W^s(\mathbf{0}) \bigcap B_1$ containing $\mathbf{0}$.

In the following we choose B_1 small enough so that the indicated statements are valid.

Let $\Gamma = W^s_{loc} \bigcap B_{1top}$. Then Γ divides B_{1top} into two connected open sets each of whose closures is a topological 2-disk. By the Grobman-Hartman theorem, if $x \in B_{1top} \setminus \Gamma$, then there exists $\tau_\phi(x) > 0$, such that $\varphi(x, \tau_\phi(x)) \in B_1^+ \cup B_1^-$, and for $0 \leq s < \tau(x)$, $\varphi(x, s)$ is not in $B_1^+ \cup B_1^-$.

Let

$$B^+_{1top} = \{x \in B_{1top} | \varphi(x, \tau_\phi(x)) \in B_1^+\},$$

and

$$B^-_{1top} = \{x \in B_{1top} | \varphi(x, \tau_\phi(x)) \in B_1^-\}.$$

Then, the map $P(x) = \varphi(x, \tau_\phi(x))$ is called a Poincaré map from $B^+_{1top} \cup B^-_{1top} \setminus \Gamma$ to the sides $B_1^+ \cup B_1^-$.

Recall that we use the maximum norm

$$|(v_1, v_2, v_3)| = \max(|v_1|, |v_2|, |v_3|)$$

on vectors $v = (v_1, v_2, v_3) \in R^3$ and its induced norm on various other quantities. For instance, the arclength of curves, lengths of tangent vectors, and norms of linear maps are all defined relative to this maximum norm.

Consider a C^2 curve γ in B_{1top} which is transverse to Γ at the point $p \in \Gamma$. Let $\gamma^{\pm} = \gamma \cap B_{1top}^{\pm}$. Assume that γ is parametrized by arclength and, for $y \in \gamma$, let ℓ_y denote the arclength of the subarc of γ from p to y. Assume that there are constants $A_2 > A_1 > 0$ such that

$$A_1 \ell_y \leq |y - p| \leq A_2 \ell_y.$$

Let v_y be the unit tangent vector to $\gamma^{\pm}$ at the point $y \in \gamma^{\pm}$, and let $P'(y) = DP_y(v_y)$.

The main result needed to prove Theorem 1.0.1 is then the following

Theorem 3.0.6. *There are constants $C_2 > C_1 > 0$ such that*

$$C_1 |P'(y)| \leq |y - p|^{|\frac{b}{a}|-1} \leq C_2 |P'(y)|.$$

3.1 A sketch of the proof

The proof of Theorem 3.0.6, which we call the *quantitative partial linearization theorem*, is divided into several parts, shown step by step in the subsequent sections.

First, we straighten $W^s_{loc}(\mathbf{0})$ and $W^u_{loc}(\mathbf{0})$ and then obtain a C^2 surface Σ in B_1 which meets B_{1top} in the curve γ and is invariant under the flow $\phi = \phi(x,t)$ on orbit pieces which stay in B_1. This allows us to transform the problem to an analogous problem for a flow $\eta = \eta(x,t)$ in R^2 with a hyperbolic critical point at $(0,0)$ with a transverse curve $\tilde{\gamma}$ through a point $\tilde{p}$ in the stable manifold of η. Now, a well-known theorem of Hartman [2] gives a C^1 linearization of η in some neighborhood of $(0, 0)$. Using this linearization, we can complete the proof of the Theorem 3.0.6.

General comment. We emphasize that, while many of the general ideas in our construction have been available in the dynamics literature, we believe that our results are new in at least two ways.

- the construction of the invariant C^2 surface Σ containing an arbitrary transverse curve γ.
- the detailed estimates of the C^1 sizes in our linearization.

Comment on the two dimensional local linearization. Following Sternberg [8] to linearize a flow, one first linearizes the time T map for some $T > 0$ and uses the so-called *integral technique*. Thus the problem reduces to finding a local C^1 linearization near a hyperbolic fixed point for a C^2 planar diffeomorphism f. This theorem is also proved in Hartman [3].

A new proof for the diffeomorphism case was given by Palis and Takens in [5]. Their method consists of first C^1 linearizing along the stable and unstable manifolds, and then contructing two transverse C^1 foliations containing these manifolds which are used to define the coordinates of the required C^1 linearization. Their proof relies on a clever use of the stable manifold theorem for the derivative map Df on the tangent bundle, and, hence involves a four dimensional local diffeomorphism. While their proof can be modified to work in our case, it seems somewhat complicated to obtain estimates of the size of the domain of the linearization and its C^1 size from that approach. Since these types of estimates are important for applications, we proceed to construct our own proof of the Hartman theorem. The only thing that we borrow from Palis and Takens is the idea of transverse foliations. Our construction of the foliations and the ensuing estimates are new.

3.2 Linear estimates in dimension two

Here let us derive some properties of a linear flow in R^2:

$$\psi_t(x) = (\psi_1, \psi_2) = (e^{at}x_1, e^{bt}x_2)$$

where $a > 0 > b$. Here in R^2, we only consider $B_1 = \{x = (x_1, x_2) \in R^2 | 0 \leq x_1 \leq \varepsilon_0, 0 \leq x_2 \leq \varepsilon_0\}$, with $B_{1top} = \{(x_1, \varepsilon_0) : 0 \leq x_1 \leq \zeta\varepsilon_0, 0 < \zeta < 1\}$, $B_{1r} = \{x \in B_1 | x_1 = \varepsilon_0\}$. Since other quadrants can be done similarly, we do not show them here. Let the point $p = (0, \varepsilon_0)$, as is stated above, for any initial condition $x = (x_1, \varepsilon_0) \in B_{1top} \setminus p$, there is a finite time $\tau(x) > 0$, such that $\psi(x, \tau(x)) = (\varepsilon_0, y_2) \in B_{1r}$. Let us name such map $P_l(x_1, \varepsilon_0) = (\varepsilon_0, y_2)$, where $x_1 \neq 0$. We want to find the size of DP_l, so we proceed as follows.

We have $P_l(x_1, \varepsilon_0) = (\varepsilon_0, y_2) = (e^{at}x_1, e^{bt}\varepsilon_0)$, where $x_1 \neq 0$. By comparing the first component, we have

$$e^t = (\frac{\varepsilon_0}{x_1})^{\frac{1}{a}}.$$

Plugging to the second component, we have

$$y_2 = \varepsilon_0 (\frac{\varepsilon_0}{x_1})^{\frac{b}{a}}.$$

So we can write

$$P_l(x_1, \varepsilon_0) = (\varepsilon_0, \varepsilon_0 (\frac{\varepsilon_0}{x_1})^{\frac{b}{a}}).$$

Since

$$\partial(\varepsilon_0 (\frac{\varepsilon_0}{x_1})^{\frac{b}{a}})/\partial x_1 = \varepsilon_0 \cdot \frac{b}{a} \cdot (\frac{\varepsilon_0}{x_1})^{\frac{b}{a}-1} \frac{-\varepsilon_0}{(x_1)^2} = -\frac{b}{a} (\frac{\varepsilon_0}{x_1})^{\frac{b}{a}+1}$$

we have

$$DP_l = \begin{pmatrix} 0 & 0 \\ \frac{\partial y_2}{\partial x_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\frac{b}{a}(\frac{\varepsilon_0}{x_1})^{\frac{b}{a}+1} & 0 \end{pmatrix}.$$

So

$$DP_l \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{b}{a}(\frac{\varepsilon_0}{x_1})^{\frac{b}{a}+1} \end{pmatrix}.$$

3.3 Conjugacy restricted to the stable manifold

In this section, we will adapt the proof of Sternberg's linearization theorem [1956] into the following quantitative version, with explicit conditions on the neighborhood and sizes of the linearization worked out.

Theorem 3.3.1. *Let $f : R \to R$ be a C^2 function such that $f(0) = 0$ and $f'(0) = a$ where $|a| \in (0, 1)$. If in $B_{\varepsilon_0}(0) = \{y \in R : |y| \leq \varepsilon_0\}$, we have $|f''| \leq M$ and $\beta = |a| + M\varepsilon_0/2 < 1$, then there is a unique C^1 map $u : B_{\varepsilon_0} \to R$, such that*

$u'(0) = 0$, *and if* $h(y) = (id + u)(y)$ *then*

$$h(f(y)) = ah(y).$$

Moreover,

$$|h| \leq \varepsilon_0 \ \exp(\frac{C}{1-\beta}) \quad and \quad |h^{'}| \leq \exp(\frac{2C}{1-\beta})$$

where $C = M\varepsilon_0/(2|a|)$.

Proof: Using Taylor's theorem, in a small neighborhood of the origin B_ε, there is a real number $\theta_y \in (0, y)$ such that

$$f(y) = ay + \frac{f^{''}(\theta_y)}{2} y^2. \tag{3.1}$$

For $k \geq 0$, let

$$f_k(y) = \underbrace{f \circ f \circ \cdots \circ f}_{k}(y)$$

so that

$$f_k(y) = y \prod_{i=1}^{k} f_i(y)/f_{i-1}(y)$$

and

$$\frac{f_k(y)}{a^k} = y \prod_{i=1}^{k} f_i(y)/(af_{i-1}(y)).$$

Since

$$f_i(y) = f(f_{i-1}(y)) = af_{i-1}(y) + \frac{f^{''}(\theta_i)}{2}(f_{i-1}(y))^2$$

we have

$$f_i(y)/(af_{i-1}(y)) = 1 + \frac{f''(\theta_i)}{2a} f_{i-1}(y),$$

and, hence,

$$\frac{f_k(y)}{a^k} = y\prod_{i=1}^{k} f_i(y)/(af_{i-1}(y)) = y\prod_{i=1}^{k} [1 + \frac{f''(\theta_i)}{2a} f_{i-1}(y)].$$

Let us define a function $h(y)$ by

$$h(y) = \lim_{k\to\infty} \frac{f_k(y)}{a^k}.$$

Assuming this, notice that it provides a linearization since

$$h(f(y)) = \lim_{k\to\infty} \frac{f_k(f(y))}{a^k} = \lim_{k\to\infty} \frac{f_{k+1}(y)}{a^k} = \lim_{k\to\infty} a\frac{f_{k+1}(y)}{a^{k+1}} = ah(y).$$

Let us proceed to prove that $h(y)$ is well defined; i.e., that the limit exists.

We have

$$h(y) = \lim_{k\to\infty} \frac{f_k(y)}{a^k} = \lim_{k\to\infty} y\prod_{i=1}^{k} [1 + \frac{f''(\theta_i)}{2a} f_{i-1}(y)] \tag{3.2}$$

so, it suffices to show that

$$\sum_{i=1}^{\infty} \mid \frac{f''(\theta_i)}{2a} f_{i-1}(y) \mid < \infty. \tag{3.3}$$

Let

$$|\cdot| = \sup_{y \in B_{\varepsilon_0}} |\cdot|.$$

Now,

$$f(y) = ay + \frac{f''(\theta)}{2} y^2 = (a + \frac{f''(\theta)}{2} y) y$$

and, since $|f''| \leq M$ and $|a| \in (0, 1)$, we have

$$|f(y)| \leq (|a| + \frac{M}{2}\varepsilon_0)|y|.$$

By induction, for $i \geq 1$ we then get

$$|f_{i-1}(y)| \leq (|a| + \frac{M}{2}\varepsilon_0)^{i-1}|y| \leq (|a| + \frac{M}{2}\varepsilon_0)^{i-1}\varepsilon_0.$$

Hence

$$|\frac{f''(\theta_i)}{2a} f_{i-1}(y)| \leq \frac{M}{2|a|}(|a| + \frac{M}{2}\varepsilon_0)^{i-1}\varepsilon_0 = C\beta^{i-1} \tag{3.4}$$

where the constants $C = \dfrac{\varepsilon_0 M}{2|a|}$ and $\beta = |a| + \frac{M}{2}\varepsilon_0 \in (0, 1)$ are those given in the theorem. The above inequality implies that the infinite product in (3.2) converges uniformly, and thus, the linearization $h(y)$ exists. As the uniform limit of continuous functions is continuous, we conclude that $h(y)$ is continuous.

Using $log(1 + x) \leq x$ for $x > 0$, we have, for any positive real numbers p_n

$$
\begin{aligned}
\prod_{n=1}^{\infty}(1+p_n) &= exp(\log(\prod_{n=1}^{\infty}(1+p_n))) = exp(\sum_{n=1}^{\infty}\log(1+p_n)) \\
&\leq exp(\sum_{n=1}^{\infty}p_n).
\end{aligned}
$$

Using that

$$
h(y) = \lim_{k\to\infty}\frac{f_k(y)}{a^k}
$$

where

$$
\frac{f_k(y)}{a^k} = y\prod_{i=1}^{k} f_i(y)/(af_{i-1}(y)) = y\prod_{i=1}^{k}[1+\frac{f''(\theta_i)}{2a}f_{i-1}(y)]
$$

we get

$$
\begin{aligned}
\mid h(y)\mid &= \mid y\left(\prod_{i=1}^{\infty}[1+\frac{f''(\theta_i)}{2a}f_{i-1}(y)]\right)\mid \\
&\leq \varepsilon_0\ exp(\sum_{i=1}^{\infty}C\beta^{i-1}) \leq \varepsilon_0\ exp(\frac{C}{1-\beta}).
\end{aligned}
$$

Further, for each $y\in(0,\varepsilon_0)$ there is a $\tau_i\in(0,\varepsilon_0)$ such that

$$
f'(y) = a + f''(\tau_i)y.
$$

Since, for each $k>0$ we have (with a possibly different τ_i)

$$\begin{aligned}\frac{f_k'(y)}{a^k} &= \prod_{i=1}^{k}(\frac{f'(f_{i-1}(y))}{a}) = \prod_{i=1}^{k}(\frac{a+f''(\tau_i)f_{i-1}(y)}{a}) \qquad (3.5)\\ &= \prod_{i=1}^{k}(1+\frac{f''(\tau_i)}{a}f_{i-1}(y))\end{aligned}$$

and, as above,

$$\mid \frac{f''(\tau_i)}{a}f_{i-1}(y)) \mid \leq 2C\ \beta^{i-1}$$

we get $h'(y)$ exists and

$$h'(y) = \lim_k \frac{f_k'(y)}{a^k}.$$

In addition, with C as above, we get

$$\mid h'(y) \mid \leq exp(\frac{2C}{1-\beta}).$$

Notice that, from the last expression in (3.5), we have that

$$\frac{f_k'(y)}{a^k} = 1$$

for $y = 0$. Thus, $h(y) = y + u(u)$ where u is a C^1 function such that $u'(0) = 0$.

By now we have shown the existence of a C^1 linearization h described in the theorem. It remains to show such h is unique. Let $L(y) = ay$. Suppose there is another function $g : B_{\varepsilon_0} \to R$ of the same properties as h; i.e., $g^{-1}fg = L$, $g(0) = 0$, and $g'(0) = 1$. Then $f = gLg^{-1}$ and so $h^{-1}gLg^{-1}h(y) = ay$. Let $\rho = h^{-1}g$, we have $\rho L \rho^{-1} = L$, $\rho(0) = 0$, and $\rho'(0) = 1$. So we can write $\rho(y) = y + v(y)$ with $v(y) \in C^1$, $v(0) = 0$ and $v'(0) = 0$. Then let us show for such ρ, we have $\rho(y) = y$ for all $y \in B_{\varepsilon_0}$, i.e., $v(y) = 0$ for all $y \in B_{\varepsilon_0}$. By $\rho L \rho^{-1}(y) = ay$, we have $\rho L(y) = a\rho(y)$, which is

$$ay + v(ay) = a(y + v(y))$$

so $v(ay) = av(y)$, for all $y \in B_{\varepsilon_0}$. Taking derivatives, we get $av'(y) = av'(ay)$, i.e., $v'(y) = v'(ay)$ for all $y \in B_{\varepsilon_0}$. We claim that $v'(y) = 0$ for all $y \in B_{\varepsilon_0}$. Suppose there is $z \in B_{\varepsilon_0}$ such that $v'(z) \neq 0$, then $v'(z) = v'(az) \neq 0$. Since $|a| < 1$, $a^i z \in B_{\varepsilon_0}$ for integer $i \geq 0$. We have

$$v'(a^n z) = \cdots = v'(az) = v'(z) \neq 0.$$

But $\lim_{n\to\infty} a^n z = 0$, by the continuity of v' we have $v'(z) = v'(0) = 0$ which is a contradiction. So the claim is proved, which says $v'(y) = 0$ for all $y \in B_{\varepsilon_0}$, i.e, $v(y)$ is a constant function. Since $v(0) = 0$, we have $v(y) = 0$, giving $\rho(y) = y$ or thus $h = g$. The uniqueness is proved, and so the proof of Theorem 3.3.1 is complete.

Now consider a local C^2 flow ϕ_t, $|t| \leq 1$, on R, with a hyperbolic critical point at the origin, i.e., if $f = \phi_1$, then f satisfies the assumptions of Theorem 3.3.1. Thus, $f(x) = ax + \tilde{f}(x)$, where $0 < a < 1$, $x \in R$, and $\tilde{f}(x)$ stands for the higher order terms. Using the uniqueness of h, which is derived from a standard technique as in [ref Newhouse Notes] show that h also linearizes $\{\phi_t\}_{|t|\leq 1}$. That is, if $\psi_t(x) = e^{at}x$, then $h\phi_t(x) = \psi_t h(x)$ for $x \in B_{\varepsilon_0}$.

3.4 Local C^1 linearization for two dimensional diffeomorphisms

Remark: *It is well-known that linearizations for flows follow from consideration of associated time−1 maps. Hence we restrict here to the case of diffeomorphisms (see p. 817 in [8]).*

In this section, we consider the C^2 diffeomorphism $f : R^2 \to R^2$ with a hyperbolic fixed point at $(0,0)$. We assume the local unstable and stable manifolds of f, denoted by $W^u(\mathbf{0}), W^s(\mathbf{0})$, are contained in the coordinate lines $\{(x_1,x_2) : x_2 = 0\}, \{(x_1,x_2) : x_1 = 0\}$, respectively.

Let $\varepsilon_0 > 0$ and consider the square

$$B_{\varepsilon_0} = \{(x_1,x_2) : |x_1| \leq \varepsilon_0, |x_2| \leq \varepsilon_0\}.$$

For $x = (x_1,x_2) \in B_{\varepsilon_0}$, we assume that

$$f(x_1,x_2) = (\lambda_1 x_1 + \tilde{f}_1(x_1,x_2), \lambda_2 x_2 + \tilde{f}_2(x_1,x_2)) = (f_1,f_2) \tag{3.6}$$

where $\tilde{f}_i$, $i = 1,2$, are the nonlinear terms, and $\lambda_1 > 1,\ 0 < \lambda_2 < 1$.

Let us write the Jacobian matrix of Df_x as

$$Df_x = \begin{pmatrix} A_x & B_x \\ C_x & D_x \end{pmatrix}.$$

In addition, we assume that there are a small $\epsilon > 0$ and a constant $K > 0$ such that

$$\tilde{f}_i(0,0) = 0 \text{ and the first partial derivatives } \tilde{f}_{ix_j}(0,0) = 0, \text{ for } i,j = 1,2 \tag{3.7}$$

$$|B| \leq \epsilon, \ |C| \leq \epsilon, \ 0 < \lambda_2 - \epsilon \leq D \leq \lambda_2 + \epsilon < 1, \ 1 < \lambda_1 - \epsilon \leq A \leq \lambda_1 + \epsilon, \tag{3.8}$$

and,

$$|dA^{-1}| \leq K, \ |dB| \leq K, \ |dC| \leq K, \ |dD| \leq K. \tag{3.9}$$

Using the previous section, suppose we have found C^1 linearizations $h^s : U \to R$ of $f \mid_{W^s(\mathbf{0})}$ and $h^u : U' \to R$ of $f \mid_{W^u(\mathbf{0})}$. Then in this section, we will proceed to linearize $f : R^2 \to R^2$ in some neighborhood B_1 of $W^s(\mathbf{0}) \bigcup W^u(\mathbf{0})$ in B_{ε_0}. This will be done via a modification of ideas in [[5]]. We will construct the neighborhood B_1 and C^1 submersions $\pi_u : B_1 \to W^s(\mathbf{0})$ and $\pi_s : B_1 \to W^u(\mathbf{0})$ which commute with f, and, together with the linearizations h^s and h^u restricted to the local stable and unstable manifolds, will give a linearization h in B_1 as

$$h(q) = (h_1, h_2) = (h^u \pi_s(q), h^s \pi_u(q)) = \bar{q}$$

$$h^{-1}(\bar{q}) = \pi_u^{-1}(h^s)^{-1}(\bar{q}_y) \cap \pi_s^{-1}(h^u)^{-1}(\bar{q}_x).$$

The inverse images of the submersions π_s, π_u will define two transverse folations which are invariant by f. These foliations, after adjustments using h^u, h^s, become the local coordinate curves of the linearization h.

We actually only deal with the construction of π_u, leaving the analogous construction of π_s to the reader.

Thus, we prove the following theorem.

Theorem 3.4.1. *Let $f : R^2 \to R^2$ be a C^2 diffeomorphism satisfying (3.6)-(3.9) above. Define*

$$\alpha = \frac{1}{(\lambda_1 - \epsilon)(\lambda_2 - \epsilon)}, \quad M = \frac{\lambda_1 + 2\epsilon}{\alpha(1 - \frac{\epsilon^2}{\alpha})}, \quad \mu = \frac{M(\lambda_2 + 2\epsilon) + \epsilon}{\lambda_1 - \epsilon}. \tag{3.10}$$

Assume that ϵ is small enough so that

$$\frac{\lambda_2 + 3\epsilon}{\lambda_1 - \epsilon} < 1 \text{ and } \mu < 1. \tag{3.11}$$

Then, there is a neighborhood B_1 of $W^s(\mathbf{0}) \bigcup W^u(\mathbf{0})$ in B_{ε_0} and a C^1 submersion $\pi_u : B_1 \to W^s(\mathbf{0})$ such that

$$\pi_u f = f\pi_u \ \mathit{on}\ B_1 \bigcap f^{-1}(B_1). \tag{3.12}$$

Moreover, one has the estimates

$$|\pi_u(x_1, x_2) - x_2| \leq \varepsilon_0, \tag{3.13}$$

and

$$\begin{aligned} |\frac{\partial \pi_u}{\partial x_1}| &\leq \varepsilon_0(1 + \frac{F}{1-\mu}) \cdot \exp(\frac{\varepsilon_0 F}{1-\mu}) \\ |\frac{\partial \pi_u}{\partial x_2} - 1| &\leq (\frac{\varepsilon_0 \cdot F}{1-\mu}) \exp(\frac{\varepsilon_0 \cdot F}{1-\mu}) \end{aligned}$$

where

$$F = M \cdot K(\frac{3}{\lambda_1 - \epsilon} + \lambda_2 + 3\epsilon).$$

Let us start proving the theorem. We write $f(x_1, x_2) = (f_1(x_1, x_2), f_2(x_1, x_2))$.

Consider the sets $B^+_{\varepsilon_0}$ and $B^-_{\varepsilon_0}$ defined by

$$B^+_{\varepsilon_0} = \{(x_1, x_2) \in B_{\varepsilon_0} | x_2 \geq 0\}, \tag{3.14}$$

$$B^-_{\varepsilon_0} = \{(x_1, x_2) \in B_{\varepsilon_0} | x_2 \leq 0\} \tag{3.15}$$

We will construct π_u on $B^+_{\varepsilon_0}$ and leave the analogous construction on $B^-_{\varepsilon_0}$ to the reader.

We choose $0 < y_0 < \varepsilon_0$, and a small $\delta > 0$, such that

$$f^{-1}(0, y_0 + \delta) \in B^+_{\varepsilon_0}.$$

If we write $p = (0, y_0)$, and $p^{\pm} = (0, y_0 \pm \delta)$, then the above condition becomes

$$f^{-1}(p^+) \in B^+_{\varepsilon_0}.$$

We write I_1 to be the interval between p^- and p^+ on the y-axis, and similarly write I_2 to be the interval between $f(p^-)$ and $f(p^+)$, and I_0 to be the interval between $f(p^-)$ and p^+ (see figure 3.1). In addition, we use I_{-1} to denote the set of the second coordinates of $f^{-1}(0 \times I_0)$ (see figure 3.1). Then we use l_0 to represent the horizontal line segment that passes through p inside $B^+_{\varepsilon_0}$. And we take D_0 as the region bounded above and below by l_0 and $f(l_0)$, which stays inside $B^+_{\varepsilon_0}$, including its boundaries (see figure 3.2).

We define B_1 to be the connected component of $\bigcup_{n=0}^{\infty}(f^n(D_0) \cap B^+_{\varepsilon_0})$ that contains p. Now let us proceed to define a C^2 horizontal foliation $\mathcal{F}^u$ over the base I_0, which can then be extended to the whole B_1 by the iteration of f.

First, we may choose a smooth bump function $\rho(x_2)$ over I_0 such that, $\rho(x_2) = 1$ over I_1; $\rho(x_2) = 0$ over I_2; and over $I_0 \setminus (I_1 \cup I_2)$, $\rho(x_2)$ is monotonically increasing w.r.t. x_2 with $0 < \rho(x_2) < 1$.

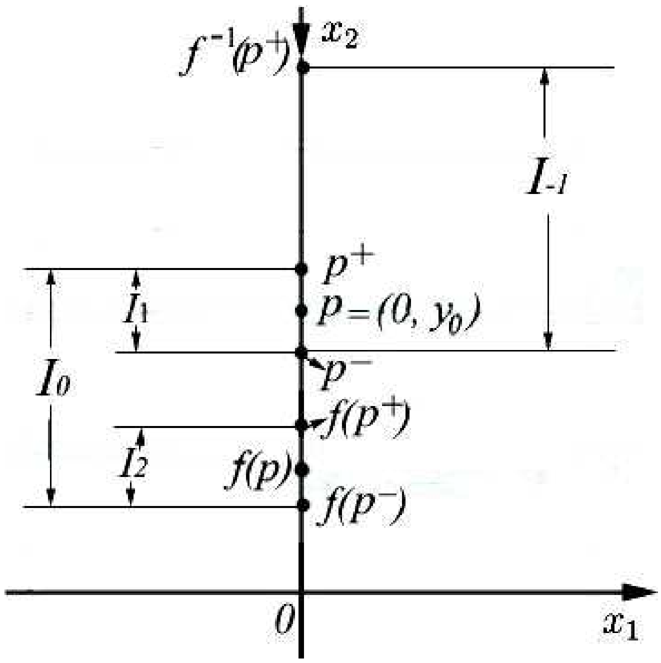

Figure 3.1: Intervals

Second, we define the first horizontal foliation $\mathcal{F}^1$ over the base $I_0 \cup I_{-1}$ (see figure 3.3). For every given $x_2 \in I_0 \cup I_{-1}$, define the leaf through $(0, x_2)$ as the set

$$\mathcal{F}^1(x_2) = \{(x_1, x_2)|\ |x_1| \le \varepsilon_0\}$$

which can also be rewritten as

$$\mathcal{F}^1(x_2) = \{(x_1, \eta_1(x_1)|\ |x_1| \le \varepsilon_0\}$$

where $\eta_1(x_1) = x_2$ for every given $x_2 \in I_0 \cup I_{-1}$, i.e., η_1 is a constant function.

Next, let us define our second horizontal foliation $\mathcal{F}^2$ (see figure 3.4) over the

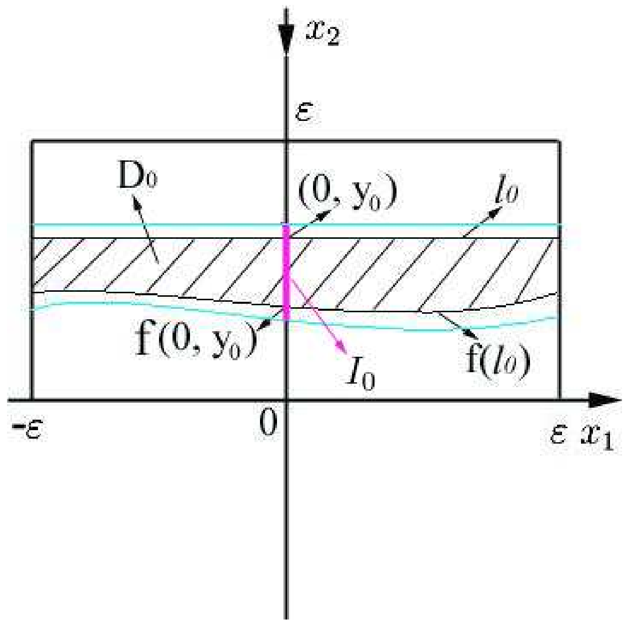

Figure 3.2: D_0. For interpretation of the references to color in this and all other figures, the reader is referred to the electronic version of this dissertation

base I_0. We write $f^{-1}(0, x_2) = (0, \tilde{x}_2) \in B^+_{\varepsilon_0}$. And for every given $x_2 \in I_0$, define the leaf through $(0, x_2)$ to be

$$\mathcal{F}^2(x_2) = f(\mathcal{F}^1(\tilde{x}_2)) \cap B^+_{\varepsilon_0}.$$

We may also write it as

$$\mathcal{F}^2(x_2) = \{(x_1, \eta_2(x_1)| \, |x_1| \leq \varepsilon_0\}$$

where $\eta_2(0) = x_2$.

Finally, our desired horizontal foliation $\mathcal{F}^u$ over I_0 is a combination of $\mathcal{F}^1$ and $\mathcal{F}^2$ via the bump function $\rho(x_2)$. As for every given $x_2 \in I_0$, we define the leaf of

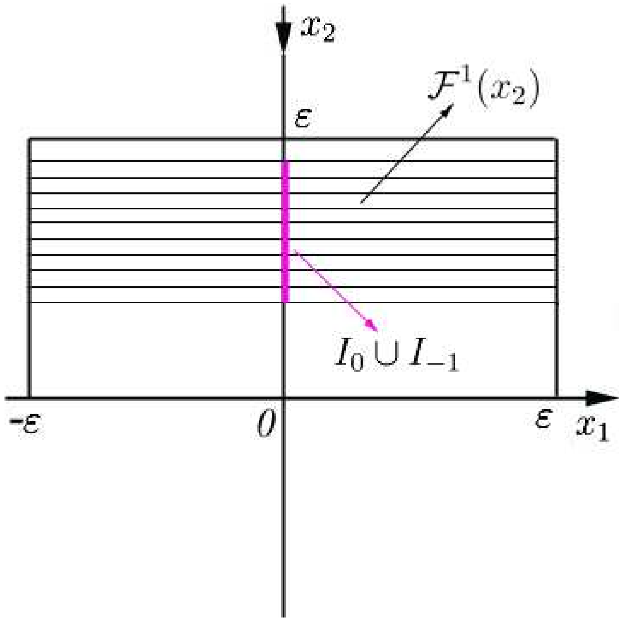

Figure 3.3: $\mathcal{F}^1(y)$

$\mathcal{F}^u$ through $(0, x_2)$ to be:

$$\mathcal{F}^u(x_2) = \{(x_1,\ \rho(x_2)\eta_1(x_1) + (1-\rho(x_2))\eta_2(x_1)) :\ |x_1| \le \varepsilon_0\}.$$

Observe that, by our definition of $\rho(x_2)$, $\mathcal{F}^u(x_2)$ agrees with $\mathcal{F}^1(x_2)$ for $x_2 \in I_1$, and agrees with $\mathcal{F}^2(x_2)$ for $x_2 \in I_2$. For every given x_2, the tangent vectors along the leaf $\mathcal{F}^1(x_2)$ equals 0; and the tangent vectors along the leaf $\mathcal{F}^2(x_2)$ can be written as

$$\begin{pmatrix} f_{1x_1} & f_{1x_2} \\ f_{2x_1} & f_{2x_2} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} f_{1x_1} \\ f_{2x_1} \end{pmatrix} \xrightarrow{rescaled} \begin{pmatrix} 1 \\ \frac{f_{2x_1}}{f_{1x_1}} \end{pmatrix}$$

where we know $f_{1x_1} \neq 0$. So the tangent vectors to the leaf $\mathcal{F}^u(x_2)$ can be repre-

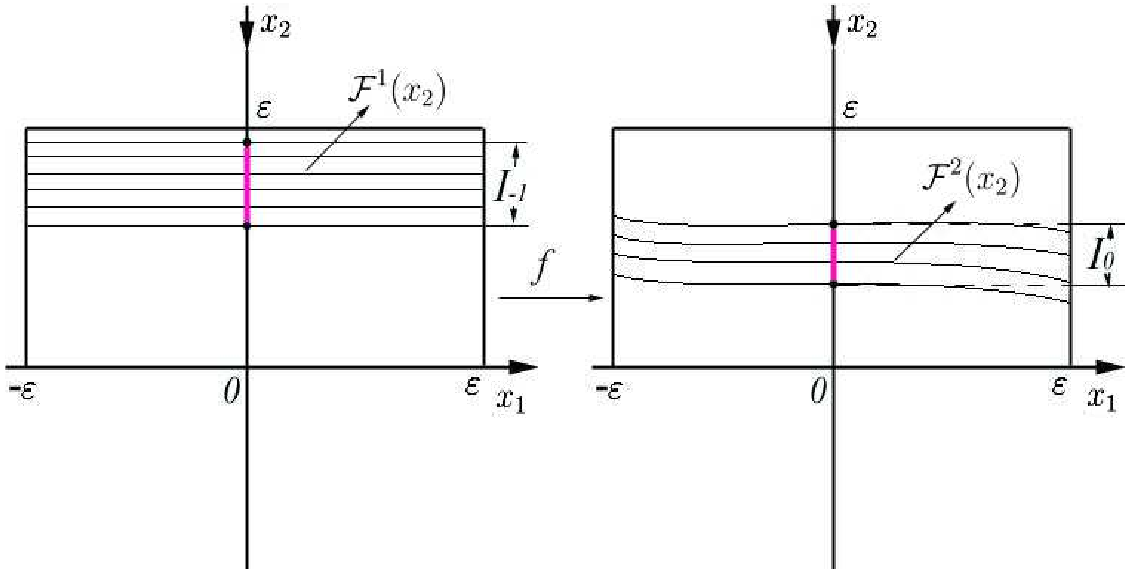

Figure 3.4: $\mathcal{F}^2(y)$

sented as

$$\rho(x_2)\begin{pmatrix}1\\0\end{pmatrix}+(1-\rho(x_2))\begin{pmatrix}1\\\frac{f_{2x_1}}{f_{1x_1}}\end{pmatrix}=\begin{pmatrix}1\\(1-\rho(x_2))\frac{f_{2x_1}}{f_{1x_1}}\end{pmatrix}. \tag{3.16}$$

The computation above also implies that our leafs of $\mathcal{F}^u(x_2)$ for different x_2's do not intersect, i.e, the tangent vector at every point to a given leaf is uniquely defined. Moreover, in the next paragraph we show the slope of each tangent vector is bounded by 1, which implies our foliation $\mathcal{F}^u(x_2)$ is well defined.

Let

$$X_u=\begin{pmatrix}X_{u,1}\\X_{u,2}\end{pmatrix}=\begin{pmatrix}1\\(1-\rho(x_2))\frac{f_{2x_1}}{f_{1x_1}}\end{pmatrix}.$$

We will need the following estimate, which says the absolute value of the slope of each tangent vector above is less than 1. Recall our assumption in (3.8), we know $f_{1x_1} \geq \lambda_1 - \epsilon > 1$ and $|f_{2x_1}| \leq \epsilon$, so we have

$$\begin{aligned}
\frac{|X_{u,2}|}{|X_{u,1}|} &= |(1-\rho(x_2))\frac{f_{2x_1}}{f_{1x_1}}| \\
&\leq |\frac{f_{2x_1}}{f_{1x_1}}| \\
&\leq \frac{\epsilon}{\lambda_1 - \epsilon} < 1.
\end{aligned}$$

We now extend X_u to B_1. For each $x \in B_1 \setminus B_{1bot}$, there is a least integer $n(x)$, such that $f^{-n(x)}(x) \in D_0$. Let

$$X_u(x) = Df^{n(x)}_{f^{-n(x)}(x)} \cdot X_u(f^{-n(x)}(x)).$$

A λ-lemma (inclination lemma) argument shows that the absolute value of the slope $|X_{u,2}(x)/X_{u,1}(x)|$ of $X_u(x)$ remains bounded by 1 for all $x \in B_1$ and converges to 0 as x approaches B_{1bot}. Let

$$\widetilde{P_x} = \frac{X_{u,2}(x)}{X_{u,1}(x)}$$

be this slope. For each x we define

$$Y_u = \begin{pmatrix} 1 \\ \widetilde{P_x} \end{pmatrix} \tag{3.17}$$

for $x \in B_1 \setminus B_{1bot}$, and

$$Y_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

for $x \in B_{1bot}$. This $\widetilde{P_x}$ is a continuous function from B_1 to R, such that $|\widetilde{P_x}| \leq 1$ for all $x \in B_1$.

Our main result of this section is the following.

Theorem 3.4.2. *The vector field Y_u constructed above is C^1 in B_1. Equivalently, the function $x \to \widetilde{P_x}$ is C^1 in B_1.*

As we indicated above the integral curves of Y_u will give our desired foliation $\mathcal{F}^u$. The proof that Y_u is C^1, or equivalently, the function $x \to \widetilde{P_x}$ is C^1, will require several steps. The proof will be obtained by finding a sequence g_1, g_2, $\cdots$, of C^1 functions, such that

1. g_n converges to $\widetilde{P}$ uniformly on B_1,

2. Dg_n converges uniformly on B_1.

Following this, it is standard that $\widetilde{P}$ is C^1 and $D\widetilde{P} = \lim_{n\to\infty} Dg_n$.

We will use the technique of the Fiber Contraction Theorem familiar from the Invariant Manifold Theory as in [4]. This involves definitions of certain function spaces and associated mappings. Let us proceed to define our first function space $\mathcal{G}$.

Consider the space $C^0(B_1, R)$ of bounded continuous functions from the box B_1 to R with supremum norm $\sup_{x\in B_1} |\cdot|$ that makes it a Banach space. Let

$C_1^0(B_1, R)$ be the closed subset of $C^0(B_1, R)$ with each of its element bounded by 1. Then $C_1^0(B_1, R)$ is a complete metric space with the metric induced by this norm. Let

$$\mathcal{G} = \{P | P \in C_1^0(B_1, R),\ P(x) = \tilde{P}(x)\ for\ x \in D_0 \cup B_{1bot}\}.$$

$\mathcal{G}$ is also a complete metric space equipped with the same norm $\sup_{x \in B_1} | \cdot |$.

The first step is to show that $\widetilde{P_x}$ is the unique fixed point of a contraction map Γ on $\mathcal{G}$, which we now define. Let $f = \phi_{X,T}$, by construction we have

$$Df y \cdot \begin{pmatrix} 1 \\ \widetilde{P_y} \end{pmatrix} = \begin{pmatrix} c \\ c \cdot \widetilde{P_x} \end{pmatrix} \tag{3.18}$$

where $y = f^{-1}(x)$ and c is a scalar. Using the notation at the beginning of this section, we write

$$\begin{pmatrix} A_y & B_y \\ C_y & D_y \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \widetilde{P_y} \end{pmatrix} = \begin{pmatrix} c \\ c \cdot \widetilde{P_x} \end{pmatrix}. \tag{3.19}$$

By solving the above equations for $\widetilde{P}$, we have

$$\begin{aligned} A_y + B_y \widetilde{P_y} &= c \\ C_y + D_y \widetilde{P_y} &= c \cdot \widetilde{P_x}. \end{aligned}$$

So

$$\widetilde{P_x} = \frac{C_y + D_y \widetilde{P_y}}{A_y + B_y \widetilde{P_y}}$$

which is equivalent to

$$A_y \widetilde{P_x} + \widetilde{P_y} B_y \widetilde{P_x} = C_y + \widetilde{P_y} D_y.$$

So we have

$$\widetilde{P_x} = A_y^{-1} C_y + A_y^{-1} D_y \widetilde{P_y} - A_y^{-1} B_y \widetilde{P_y} \widetilde{P_x}.$$

Given $P \in \mathcal{G}$, we define $\Gamma(P)$ as follows. For $x \in D_0 \cup B_{1bot}$, set

$$\Gamma(P)_x = \widetilde{P_x}.$$

For $x \in B_1 \setminus (D_0 \cup B_{1bot})$, let $y = f^{-1}(x)$, and

$$\Gamma(P)_x = A_y^{-1} C_y + A_y^{-1} D_y P_y - A_y^{-1} B_y P_x P_y. \tag{3.20}$$

Clearly, $\Gamma(P)_x$ is continuous in x. To show that $\Gamma(P) \in \mathcal{G}$, it suffices to show that

$$\sup_{x \in B_1} |\Gamma(P)_x| \leq 1.$$

For each $x \in B_1$ and $y = f^{-1}(x)$, we have

$$\begin{aligned} |\Gamma(P)_x| &\leq |A_y^{-1} C_y| + |A_y^{-1} D_y P_y| + |A_y^{-1} B_y P_x P_y| \\ &\leq \frac{\epsilon}{\lambda_1 - \epsilon} + \frac{\lambda_2 + \epsilon}{\lambda_1 - \epsilon} + \frac{\epsilon}{\lambda_1 - \epsilon}. \end{aligned}$$

If the size of B_1 is properly chosen, we will have

$$\frac{\lambda_2 + 3\epsilon}{\lambda_1 - \epsilon} < 1.$$

So

$$|\Gamma(P)| < 1$$

which means Γ maps $\mathcal{G}$ into itself.

Moreover, the Lipschitz constant λ of Γ can be computed as follows

$$\begin{aligned} |\Gamma(P_1)x - \Gamma(P_2)x| &\leq |A_y^{-1}C_y + A_y^{-1}D_yP_1(y) - A_y^{-1}B_yP_1(x)P_1(y) \\ &\quad -A_y^{-1}C_y - A_y^{-1}D_yP_2(y) + A_y^{-1}B_yP_2(x)P_2(y)| \\ &\leq |A_y^{-1}D_y||P_1(y) - P_2(y)| \\ &\quad +|A_y^{-1}B_y||P_1(x)P_1(y) - P_2(x)P_2(y)| \end{aligned}$$

where

$$\begin{aligned} |P_1(x)P_1(y) - P_2(x)P_2(y)| &\leq |P_1(x)P_1(y) - P_1(x)P_2(y) \\ &\quad +P_1(x)P_2(y) - P_2(x)P_2(y)| \\ &\leq |P_1(x)||P_1(y) - P_2(y)| + |P_2(y)||P_1(x) - P_2(x)|. \end{aligned}$$

Since $|P_1| \leq 1$ and $|P_2| \leq 1$,

$$|P_1(x)P_1(y) - P_2(x)P_2(y)| \leq 2|P_1 - P_2|$$

we have

$$|\Gamma(P_1) - \Gamma(P_2)| \leq (|A^{-1}D| + 2|A^{-1}B|)|P_1 - P_2|$$

Let

$$\lambda = |A^{-1}D| + 2|A^{-1}B| \leq \frac{\lambda_2 + \epsilon}{\lambda_1 - \epsilon} + 2\frac{\epsilon}{\lambda_1 - \epsilon} = \frac{\lambda_2 + 3\epsilon}{\lambda_1 - \epsilon}$$

If in B_1, we have $\lambda < 1$, then Γ is a contraction mapping with contraction constant λ. That means, for any initial choice of $P \in \mathcal{G}$, $\Gamma^n(P)$ converges uniformly to a unique fixed point. Since $\Gamma(\widetilde{P}) = \widetilde{P}$, we know $\widetilde{P}$ is the unique attracting fixed point. So we may choose our converging sequence as follows.

To get our initial P_0, we may use a smooth bump function $\rho : R^2 \to R$ such that $\rho(x) = 1$ on a neighborhood of D_0 and $\rho(x) = 0$ on a neighborhood of B_{1bot}. Then we define a C^1 function g_0, such that $g_0|_{B_1} \in \mathcal{G}$:

$$g_0 = \rho(x)\tilde{P} + (1 - \rho(x)) \cdot 0 = \rho(x)\tilde{P}$$

for all points in a neighborhood U_0 of B_1. Let $P_0 = g_0|_{B_1}$, we see that $P_0|_{D_0} = \tilde{P}$, $P_0|_{B_{1bot}} = 0$ and is C^1 on B_1. Now for $y = f^{-1}(x)$, let

$$g_1(x) = \Gamma(g_0) = A_y^{-1}C_y + A_y^{-1}D_y g_0(y) - A_y^{-1}B_y g_0(x)g_0(y)$$

which is C^1 on a neighborhood of B_1. Inductively, set

$$g_n(x) = \Gamma(g_{n-1}) = A_y^{-1}C_y + A_y^{-1}D_y g_{n-1}(y) - A_y^{-1}B_y g_{n-1}(x)g_{n-1}(y)$$

This gives us a sequence of C^1 functions g_n with the following properties:

1. g_n is defined and is C^1 on a neighborhood U_n of B_1,
2. $g_n|_{B_1} \in \mathcal{G}$,
3. $\Gamma(g_n)$ converges uniformly to $\widetilde{P}$ on B_1.

Let $P_n = g_n|_{B_1}$ for $n \geq 0$, we have $\Gamma^n(P_0) = P_n$, and we can find the C^0 size of $\widetilde{P}$, that is

$$|\widetilde{P}| \leq \frac{\lambda^n}{1-\lambda}|P_1 - P_0| + |P_n|. \tag{3.21}$$

Also it can be shown that

$$|\widetilde{P}| \leq \frac{\lambda}{1-\lambda}|P_1 - P_0|.$$

We wish to show that DP_n converges uniformly on B_1. For this purpose we will use the Fiber Contraction Theorem on a suitable space $\mathcal{G} \times \mathcal{H}$ which we proceed to define. First of all, let us briefly review some definitions and the Fiber Contraction Theorem.

Definition 3.4.3. *Let $(\mathcal{G}, d_{\mathcal{G}})$ and $(\mathcal{H}, d_{\mathcal{H}})$ be metric spaces. The map $\Lambda : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ of the form $\Lambda(P, Q) = (\Gamma(P), \Psi(P, Q))$ is called a bundle map on $\mathcal{G} \times \mathcal{H}$ over the base $\Gamma : \mathcal{G} \to \mathcal{G}$ with principle part $\Psi : \mathcal{G} \times \mathcal{H} \to \mathcal{H}$.*

Definition 3.4.4. *The bundle map is called a fiber contraction on $\mathcal{G} \times \mathcal{H}$ if there is a $k \in [0, 1)$, such that, for every $P \in \mathcal{G}$, the map $Q \to \Psi(P, Q)$ is a contraction mapping with contraction constant k.*

Theorem 3.4.5. [Fiber contraction theorem] *Let $(\mathcal{G}, d_{\mathcal{G}})$ and $(\mathcal{H}, d_{\mathcal{H}})$ be metric spaces and Λ a continuous fiber contraction on $\mathcal{G} \times \mathcal{H}$ over the base $\Gamma : \mathcal{G} \to \mathcal{G}$ with principle part $\Psi : \mathcal{G} \times \mathcal{H} \to \mathcal{H}$. If $\widetilde{P}$ and $\widetilde{Q}$ are unique attracting fixed points of Γ and $Q \to \Psi(\widetilde{P}, Q)$ respectively, then $(\widetilde{P}, \widetilde{Q})$ is a unique attracting fixed point of Λ.*

Now let us define the set of candidates of DP where P is C^1, and then construct an operator which has a unique fixed point as DP in this set. Let $\mathcal{H} = C^0(B_1, L(R^2, R))$ be the space of bounded continuous functions from B_1 to the linear maps from R^2 to R. Let Q be any element in $\mathcal{H}$, $\mathcal{H}$ is a Banach space equipped with the norm

$$|Q| = \sup_{x \in B_1, v \in R^2, |v|=1} |Q_x \cdot v|.$$

We now define an operator $Q \to \Psi(P, Q)$ for $Q \in \mathcal{H}$, $P \in \mathcal{G}$. For a C^1 function P, we have

$$\begin{aligned} d\Gamma(P)_x &= d(A^{-1}_{f^{-1}(x)} C_{f^{-1}(x)}) + d(A^{-1}_{f^{-1}(x)} D_{f^{-1}(x)} P_{f^{-1}(x)}) \\ &\quad - d(A^{-1}_{f^{-1}(x)} B_{f^{-1}(x)} P_x P_{f^{-1}(x)}). \end{aligned}$$

Let $y = f^{-1}(x)$ as before,

$$\begin{aligned} d\Gamma(P)_x &= \{dA_y^{-1} df^{-1}(x) C_y + A_y^{-1} dC_y df^{-1}(x) + d(A_y^{-1} D_y) df^{-1}(x) P_y \\ &\quad + A_y^{-1} D_y dP_y df^{-1}(x) - d(A_y^{-1} B_y) df^{-1}(x)(P_x P_y)\} \\ &\quad - (A_y^{-1} B_y)(dP_x P_y + P_x dP_y df^{-1}(x)). \end{aligned}$$

In this form, we define

$$\begin{aligned}
\Psi(P,Q)(x) &= \{dA_y^{-1}df^{-1}(x)C_y + A_y^{-1}dC_y df^{-1}(x) \\
&\quad + d(A_y^{-1}D_y)df^{-1}(x)P_y + A_y^{-1}D_y Q_y df^{-1}(x) \\
&\quad - d(A_y^{-1}B_y)df^{-1}(x)(P_x P_y)\} \\
&\quad -(A_y^{-1}B_y)(Q_x P_y + P_x Q_y df^{-1}(x)).
\end{aligned} \tag{3.22}$$

Consider the bundle map $\Lambda(P,Q) : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ defined by $\Lambda(P,Q) = (\Gamma(P), \Psi(P,Q))$. Observe that, since $\Gamma^n(P_0) = P_n$, we have

$$\Lambda(P_0, dP_0) = (\Gamma(P_0), \Psi(P_0, dP_0)) = (\Gamma(P_0), d\Gamma(P_0)) = (P_1, dP_1).$$

Suppose

$$\Lambda^{n-1}(P_0, dP_0) = (P_{n-1}, dP_{n-1})$$

Inductively,

$$\begin{aligned}
\Lambda^n(P_0, dP_0) &= (\Gamma(P_{n-1}), \Psi(P_{n-1}, dP_{n-1})) = (\Gamma(P_{n-1}), d\Gamma(P_{n-1})) \\
&= (P_n, dP_n).
\end{aligned}$$

which converges uniformly if Λ is a fiber contraction, by the Fiber Contraction Theorem. And thus dP_n converges uniformly. Now let us show that Λ is a fiber contraction.

First, we prove that for any given $P \in \mathcal{G}$, the map $Q \to \Psi(P,Q)$ maps $\mathcal{H}$ to itself. It is easy to see that $\Psi(P,Q)$, for a given P, is continuous with respect to $x \in B_1$, since f is C^2. We will need to show $\Psi(P,Q)$ is bounded. We proceed as follows

$$
\begin{aligned}
|\Psi(P,Q)| \;\leq\;& \{|dA^{-1}||C| + |A^{-1}||dC| + |d(A^{-1}D)||P| - |d(A^{-1}B)|P|^2\}|df^{-1}| \\
& + |A^{-1}D \cdot Q||df^{-1}| + |A^{-1}B||Q \cdot P + P \cdot Q \cdot df^{-1}| \\
\leq\;& \{|dA^{-1}||C| + |A^{-1}||dC| + |d(A^{-1}D)| + |d(A^{-1}B)|\}|df^{-1}| \\
& + |A^{-1}D||Q||df^{-1}| + |A^{-1}B|(|Q| + |Q||df^{-1}|) \\
\leq\;& \{|dA^{-1}||C| + |A^{-1}||dC| + |d(A^{-1}D)| + |d(A^{-1}B)|\}|df^{-1}| \qquad (3.23) \\
& + \{|A^{-1}D||df^{-1}| + |A^{-1}B|(1 + |df^{-1}|)\}|Q|
\end{aligned}
$$

where we have used $|P| \leq 1$. Now, we need to compute $|df^{-1}|$ to proceed. For

$$
df = \begin{pmatrix} A & B \\ C & D \end{pmatrix}
$$

we know

$$
df^{-1} = \frac{1}{AD - BC} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}.
$$

Recall the definitions of α and M from (3.10), we have

$$
\alpha = \frac{1}{(\lambda_1 - \epsilon)(\lambda_2 - \epsilon)} \tag{3.24}
$$

and

$$M = \frac{\lambda_1 + 2\epsilon}{\alpha(1 - \frac{\epsilon^2}{\alpha})} \approx \frac{\lambda_1}{\lambda_1 \lambda_2} = \frac{1}{\lambda_2}, \tag{3.25}$$

We claim that

$$|df^{-1}| \leq M. \tag{3.26}$$

Indeed, since $|AD| > \alpha$ and $|\frac{BC}{AD}| < \frac{\epsilon^2}{\alpha}$, we have

$$\begin{aligned}
|df^{-1}| &\leq |\frac{1}{AD - BC}|| \begin{pmatrix} D & -B \\ -C & A \end{pmatrix} | \\
&\leq |\frac{1}{AD - BC}|(|A| + |C|) \\
&\leq \frac{1}{|AD||1 - \frac{BC}{AD}|}(|A| + |C|) \\
&\leq \frac{1}{|AD||1 - \frac{BC}{AD}|}(\lambda_1 + \epsilon + \epsilon) \\
&\leq \frac{1}{\alpha(1 - \frac{\epsilon^2}{\alpha})}(\lambda_1 + 2\epsilon) = M.
\end{aligned}$$

In addition, from (3.9) we have

$$\begin{aligned}
&\{|dA^{-1}||C| + |A^{-1}||dC| + |d(A^{-1}D)| + |d(A^{-1}B)|\}|df^{-1}| \\
\leq\ & M(\epsilon K + \frac{K}{\lambda_1 - \epsilon} + K(\lambda_2 + \epsilon) + \frac{K}{\lambda_1 - \epsilon} + K\epsilon + \frac{K}{\lambda_1 - \epsilon}) \\
=\ & M \cdot K(\frac{3}{\lambda_1 - \epsilon} + \lambda_2 + 3\epsilon).
\end{aligned}$$

To simplify the expression, let us write

$$F = M \cdot K(\frac{3}{\lambda_1 - \epsilon} + \lambda_2 + 3\epsilon). \tag{3.27}$$

Now, going back to (3.23), we have

$$|\Psi(P,Q)| \leq F + (M\frac{\lambda_2 + \epsilon}{\lambda_1 - \epsilon} + \frac{\epsilon}{\lambda_1 - \epsilon}(1+M))|Q|.$$

Since $Q \in \mathcal{H}$ is also bounded, we have $\Psi(P,Q)$ is bounded. So we have proved the map $Q \to \Psi(P,Q)$ maps $\mathcal{H}$ to itself.

Next, we will show that the map $Q \to \Psi(P,Q)$ is a contraction mapping. For any given $P \in \mathcal{G}$ and any Q, $\widetilde{Q} \in \mathcal{H}$, we know

$$\begin{aligned}
\Psi(P,Q)_x - \Psi(P,\widetilde{Q})_x &= A_y^{-1} D_y Q_y df^{-1}(x) - A_y^{-1} D_y \widetilde{Q}_y df^{-1}(x) \\
&\quad -(A_y^{-1} B_y) \cdot (Q_x \cdot P_y + P_x \cdot Q_y \cdot df^{-1}(x)) \\
&\quad +(A_y^{-1} B_y) \cdot (\widetilde{Q}_x \cdot P_y + P_x \cdot \widetilde{Q}_y \cdot df^{-1}(x)) \\
&= A_y^{-1} D_y (Q_y - \widetilde{Q}_y) df^{-1}(x) - A_y^{-1} B_y (Q_x - \widetilde{Q}_y) P_y \\
&\quad -A_y^{-1} B_y P_x (Q_y - \widetilde{Q}_y) df^{-1}(x).
\end{aligned}$$

Since $|P| \leq 1$, we have

$$
\begin{aligned}
|\Psi(P,Q) - \Psi(P,\widetilde{Q})| &\leq M\frac{\lambda_2+\epsilon}{\lambda_1-\epsilon}|Q-\widetilde{Q}| + \frac{\epsilon}{\lambda_1-\epsilon}|Q-\widetilde{Q}| \\
&\quad + M\frac{\epsilon}{\lambda_1-\epsilon}|Q-\widetilde{Q}| \\
&= \underbrace{\frac{M(\lambda_2+2\epsilon)+\epsilon}{\lambda_1-\epsilon}}_{\mu}|Q-\widetilde{Q}|.
\end{aligned}
$$

Plugging in $M \approx \dfrac{1}{\lambda_2}$ from (3.25), we have

$$0 < \mu \approx \frac{1}{\lambda_1} < 1. \tag{3.28}$$

Assuming the contraction μ, we have proved that, for any given $P \in \mathcal{G}$, the map $Q \to \Psi(P,Q)$ is a contraction mapping. This implies that, for $\widetilde{P} \in \mathcal{G}$, the map $Q \to \Psi(\widetilde{P},Q)$ has a unique fixed point $\widetilde{Q} \in \mathcal{H}$. If we choose $\Lambda(P,Q) = (\Gamma(P), \Psi(P,Q))$ as the bundle map on $\mathcal{G} \times \mathcal{H}$. Then, by definition, $\Lambda(P,Q)$ is a fiber contraction on $\mathcal{G} \times \mathcal{H}$. Since it has been proved that $\widetilde{P}$ is the unique attracting fixed point of Γ, then by the Fiber Contraction Theorem, $(\widetilde{P},\widetilde{Q})$ is the unique globally attracting fixed point of Λ. It remains to show that $\widetilde{Q} = d\widetilde{P}$, so that $\widetilde{P} \in C^1$.

We may start with (P_0, dP_0), where $P_0 \in C^1$. Let $\Gamma^n(P_0) = P_n$, we have

$$\Lambda(P_0, dP_0) = (\Gamma(P_0), \Psi(P_0, dP_0)) = (\Gamma(P_0), d\Gamma(P_0)) = (P_1, dP_1).$$

Suppose

$$\Lambda^{n-1}(P_0, dP_0) = (P_{n-1}, dP_{n-1}).$$

Then

$$\begin{aligned}\Lambda^n(P_0, dP_0) &= (\Gamma(P_{n-1}), \Psi(P_{n-1}, dP_{n-1})) = (\Gamma(P_{n-1}), d\Gamma(P_{n-1})) \\ &= (P_n, dP_n).\end{aligned}$$

We know P_n converges to $\widetilde{P}$ uniformly, and dP_n converges to $\widetilde{Q}$ uniformly. It is standard that $\widetilde{Q} = d\widetilde{P}$, and thus, $\widetilde{P}$ is C^1. Now we proceed to find the size of $d\widetilde{P}$. Since

$$\begin{aligned}d\Gamma(P)_x &= \{dA_y^{-1}df^{-1}(x)C_y + A_y^{-1}dC_y df^{-1}(x) + d(A_y^{-1}D_y)df^{-1}(x)P_y \\ &\quad -d(A_y^{-1}B_y)df^{-1}(x)P_xP_y\} + A_y^{-1}D_y dP_y df^{-1}(x) \\ &\quad -(A_y^{-1}B_y)\cdot(dP_x \cdot P_y + P_x \cdot dP_y \cdot df^{-1}(x)).\end{aligned}$$

In addition, we have $|P| \leq 1$ and $|df^{-1}| \leq M$, so we can write the form of the size of $|d\Gamma(P)|$ for any $P \in \mathcal{G}$ as follows

$$\begin{aligned}|d\Gamma(P)| &\leq \overbrace{\{|dA^{-1}||C| + |A^{-1}||dC| + |d(A^{-1}D)||P| + |d(A^{-1}B)||P|^2\}M}^{F} \\ &\quad +M|A^{-1}D||dP| + |A^{-1}B|(|dP| + M|dP|) \\ &\leq F + (M|A^{-1}D| + (1+M)|A^{-1}B|)|dP| \\ &\leq F + (\tau_1\tau_2 M + (1+M)\tau_{1\varepsilon_0})|dP| \\ &= F + \mu|dP|\end{aligned}$$

where

$$F = M \cdot K(\frac{3}{\lambda_1 - \epsilon} + \lambda_2 + 3\epsilon) \leq M \cdot K(4 + 3\epsilon). \tag{3.29}$$

For $m > n$ we have

$$\begin{aligned}
|dP_m| &= |d\Gamma(P_{m-1})| \qquad (3.30)\\
&\leq F + \mu|dP_{m-1}|\\
&\leq F + \mu(F + \mu|dP_{m-2}|)\\
&= F + \mu F + \mu^2|dP_{m-2}|\\
&\cdots\\
&\leq F\sum_{i=0}^{m-n-1}\mu^i + \mu^{m-n}|dP_n|.
\end{aligned}$$

So

$$|d\widetilde{P}| = \lim_{m\to\infty}|dP_m| \leq \frac{F}{1-\mu} + \lim_{m\to\infty}\mu^{m-n}|dP_n| = \frac{F}{1-\mu}. \tag{3.31}$$

Therefore, not only have we proved that $\widetilde{P} \in C^1$, but also found the explicit formula for the C^0 and C^1 sizes of $\widetilde{P}$ which can be implemented in a computer.

Our next task is to find the C^0 and C^1 sizes of π_u, which projects B_1 to the x_2-axis along the horizontal foliation $\mathcal{F}^u$. Recall that $\mathcal{F}^u$ consists of the integral curves of the vector field defined in (3.17), i.e.,

$$Y_u = \begin{pmatrix} 1 \\ \widetilde{P_x} \end{pmatrix} = \begin{pmatrix} Y_{u,1} \\ Y_{u,2} \end{pmatrix}.$$

Let $x = (x_1, x_2)$, and $\phi(x,t)$ be the solution of the vector field Y_u with initial value $x \in B_1$, and let $\tau(x)$ be the real number such that the first coordinate of $\phi(x,\tau(x))$ is zero. Of course, if $x_1 > 0$, then $\tau(x) < 0$ and if $x_1 < 0$, then $\tau(x) > 0$. For simplicity, let us assume that $x_1 < 0$. Since the first coordinate of Y_u is 1 by definition, the maximum time $\tau(x)$ for any $x \in B_1$ is ε_0.

So we can write π_u as

$$(0, \pi_u(x)) = \phi(x, \tau(x)) = (\phi_1(x), \phi_2(x))$$

with

$$\phi(x,t) = \begin{pmatrix} x_1 + \int_0^t Y_{u,1}(\phi_s(x))ds \\ x_2 + \int_0^t Y_{u,2}(\phi_s(x))ds \end{pmatrix} = \begin{pmatrix} x_1 + t \\ x_2 + \int_0^t \widetilde{P_x} ds \end{pmatrix} = \begin{pmatrix} \phi_1(x,t) \\ \phi_2(x,t) \end{pmatrix} \tag{3.32}$$

giving

$$\begin{pmatrix} 0 \\ \pi_u(x) \end{pmatrix} = \begin{pmatrix} x_1 + \tau(x) \\ x_2 + \int_0^{\tau(x)} \widetilde{P_x} dt \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 + \int_0^{-x_1} \widetilde{P_x} dt. \end{pmatrix}.$$

We have

$$|\pi_u(x) - x_2| = |\int_0^{-x_1} \widetilde{P_x} dt| \leq |x_1| \cdot |\widetilde{P}| \leq \varepsilon_0,$$

$$\frac{\partial \phi_1}{\partial x_1} = 1,\ \frac{\partial \phi_1}{\partial x_2} = 0,\ \frac{\partial \phi_1}{\partial x_2} = 0,\ \frac{\partial \phi_2}{\partial x_j} = \frac{\partial \pi u}{\partial x_j} \text{ for } j = 1, 2,$$

$$\tau(x) = -x_1,\ \partial\tau(x)/\partial x_1 = -1,\ \text{and},\ \partial\tau(x)/\partial x_2 = 0.$$

Hence, by (3.31) and the Gronwall inequality, we have

$$\begin{aligned}
|\frac{\partial \pi u}{\partial x_1}(x)| &= |\int_0^{-x_1} D\widetilde{P}(\phi(x,\tau(x)))\cdot(\frac{\partial \phi_1}{\partial x_1}, \frac{\partial \phi_2}{\partial x_1})ds \\
&\quad + \frac{\partial \tau}{\partial x_1}(x)\widetilde{P}(\phi(x,\tau(x)))| \\
&= |\int_0^{-x_1} (\frac{\partial \widetilde{P}}{\partial x_1} + \frac{\partial \widetilde{P}}{\partial x_2}\frac{\partial \pi u}{\partial x_1})ds - \widetilde{P}(\phi(x,\tau(x)))| \qquad (3.33) \\
&\leq |\widetilde{P}| + \frac{\varepsilon_0 F}{1-\mu} + \int_0^{|x_1|} \left(\frac{F}{1-\mu}\right) |\frac{\partial \pi u}{\partial x_1}| ds \\
&\leq \varepsilon_0 (1 + \frac{F}{1-\mu}) exp(\frac{\varepsilon_0 F}{1-\mu}),
\end{aligned}$$

when $|\widetilde{P}| \leq \varepsilon_0$ in small B_1, which is possible due to the Inclination Lemma, as we have $|\widetilde{P}| = 0$ along our initial B_{1top}. And we have

$$
\begin{aligned}
&|\frac{\partial \pi u}{\partial x_2}(x) - 1| \\
= \ & |\int_0^{\tau(x)} D\widetilde{P}(\phi(x,\tau(x))) \cdot (\frac{\partial \phi_1}{\partial x_2}, \frac{\partial \phi_2}{\partial x_2}) ds + \frac{\partial \tau}{\partial x_2}(x)\widetilde{P}(\phi(x,\tau(x)))| \\
= \ & |\int_0^{\tau(x)} \frac{\partial \widetilde{P}}{\partial x_2}(\frac{\partial \pi u}{\partial x_2} - 1) ds + \int_0^{\tau(x)} \frac{\partial \widetilde{P}}{\partial x_2} ds| \qquad (3.34) \\
\leq \ & \frac{\varepsilon_0 F}{1-\mu} + \int_0^{|x_1|} (\frac{F}{1-\mu}) |\frac{\partial \pi u}{\partial x_2}(x) - 1| ds \qquad (3.35) \\
\leq \ & \frac{\varepsilon_0 F}{1-\mu} exp(\frac{\varepsilon_0 F}{1-\mu}). \qquad (3.36)
\end{aligned}
$$

This completes the proof of Theorem 3.4.1.

3.5 Three dimensional vector fields

Let us recall our assumptions on the vector field introduced at the beginning. We write $x = (x_1, x_2, x_3) \in R^3$, and $X(x)$ is a forward complete C^2 vector field :

$$X(x) = (X_1(x), X_2(x), X_3(x)) = (ax_1 + \bar{X}_1(x), bx_2 + \bar{X}_2(x), cx_3 + \bar{X}_3(x))$$

where

$$c < 2b < 0 < a, \tag{3.37}$$

and

$$\bar{X}_{i,x_j}(0,0,0) = 0 \text{ for } i, j = 1, 2, 3. \tag{3.38}$$

Let

$$B_1 = \{x \in R^3 | 0 \le x_2 \le \varepsilon_0, |x_1| \le \varepsilon_0, |x_3| \le \varepsilon_0\}$$

and let $|\cdot| = \sup_{x \in B_1} |\cdot|$. Then we have a finite constant $M > 0$ such that $|\bar{X}_{i,x_j,x_k}(x)| \le M$ for $i, j, k = 1, 2, 3$. By the mean value theorem, we have some small constants $\epsilon_1 > 0$ and $\epsilon_2 > 0$, such that, $|\bar{X}_{i,x_j}(x)| \le M \cdot \varepsilon_0 = \epsilon_2$, and $|\bar{X}_i(x)| \le \epsilon_2 \cdot \varepsilon_0 = \epsilon_1$ for $i, j = 1, 2, 3$.

Let

$$B_{1top} = \{x \in B_1 | x_2 = \varepsilon_0\},$$

$$B_1^+ = \{x \in B_1 | x_1 = \varepsilon_0\},$$

$$B_1^- = \{x \in B_1 | x_1 = -\varepsilon_0\}.$$

Letting $\varphi_t(x)$ be the local flow of X, so we may write

$$\varphi(x,t) = (\varphi_1, \varphi_2, \varphi_3) = (e^{at}x_1 + \bar{\varphi}_1, e^{bt}x_2 + \bar{\varphi}_2, e^{ct}x_3 + \bar{\varphi}_3)$$

where $\bar{\varphi}_i$ for $i = 1, 2, 3$ are the higher order terms. Set $\psi_t(x)$ to be the flow of the corresponding linear vector field of X, i.e,

$$\psi_t(x) = (e^{at}x_1, e^{bt}x_2, e^{ct}x_3)$$

Let f be the time-T map φ_T, for some $0 < T \leq 1$. So $f : R^3 \to R^3$ is C^2 with a hyperbolic fixed point at $\mathbf{0}$. Let $L = Df(\mathbf{0})$, and $R^3 = E^u \oplus E^s \oplus E^{ss}$ be the splitting given by the eigenspaces of L. For convenience, we also write $L = (L^u, L^s, L^{ss}) = (L^c, L^{ss})$ with respect to the splitting $E^c \oplus E^{ss}$, where $E^c = E^u \oplus E^s$ and E^u, E^s and E^{ss} correspond to the x_1, x_2 and x_3 axes respectively. In addition, we assume that the local unstable and stable manifolds of f have been straightened, i.e., $W^u_{loc}(\mathbf{0})$ is in the x_1-axis and $W^s_{loc}(\mathbf{0})$ is in the x_2x_3-plane inside B_1.

Let

$$B_{1\varepsilon_0} = \{x \in B_1 : x_3 = 0\}$$

$$B_{2\varepsilon_0} = \{x \in B_1 : x_1 = 0, x_2 = 0\}.$$

For the most part of this section, unless we specify the index $i = 1, 2, 3$, we will always use this representation $x = (x_1, x_2) \in B_{1\varepsilon_0}$, and $y = x_3 \in B_{2\varepsilon_0}$, so that,

$$f(x, y) = (L^c x + \tilde{f}_1(x, y), L^{ss} y + \tilde{f}_2(x, y)) = (f_1(x, y), f_2(x, y)) \tag{3.39}$$

where $\tilde{f}_1(x, y)$ and $\tilde{f}_2(x, y)$ are the nonlinear terms. In B_1, we assume $f = (f_1, f_2)$ satisfies these conditions for $\lambda_1 > 1$ and $0 < \lambda_2, \lambda_3 < 1$:

$$\begin{aligned} \frac{\partial f_1}{\partial x}(\mathbf{0}) &= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \\ 0 < \frac{\partial f_2}{\partial y}(\mathbf{0}) &= \lambda_3 < (\lambda_2)^2 < 1 \end{aligned} \tag{3.40}$$

for $i = 1, 2$

$$\tilde{f}_i(\mathbf{0}) = 0 \text{ and the first partial derivatives } \tilde{f}_{ix}(\mathbf{0}) = \tilde{f}_{iy}(\mathbf{0}) = 0, \tag{3.41}$$

and

$$|\tilde{f}_{ix}| \leq \epsilon, |\tilde{f}_{iy}| \leq \epsilon, |\tilde{f}_{ixx}| \leq M_1, |\tilde{f}_{ixy}| \leq M_1, |\tilde{f}_{iyy}| \leq M_1. \tag{3.42}$$

In this section, we consider a short initial line segment in the cone of angle less than $\frac{\pi}{4}$ about the x_1-axis on the plane $x_2 = \varepsilon_0$.

Choose small positive numbers w, ζ satisfying

$$0 < w < \zeta < \frac{\varepsilon_0}{2}, \tag{3.43}$$

and, consider the line segment

$$\ell : l(s) = (l_1, l_2, l_3) = l(\mathbf{0}) + sv \subset B_1$$

where $l(\mathbf{0}) = (0, \varepsilon_0, w)$,

$$-\zeta < s < \zeta, \tag{3.44}$$

$$v = \begin{pmatrix} 1 \\ 0 \\ \theta \end{pmatrix}, \tag{3.45}$$

and $|\theta| < 1$.

Let $D_0 = \bigcup_{0 \leq t \leq T} \varphi_t(l) \subset int(B_1)$ which is possible when ℓ is short (i.e., ζ is small). And let D_1 be the connected component of $f(D_0) \cap B_1$. Inductively, let D_n be the connected component of $f(D_{n-1}) \cap B_1$. Define $\Sigma = \bigcup_{j=0}^{\infty} D_j$, and let $\Pi(x_1, x_2, x_3) = (x_1, x_2)$ be the projection onto the x_1x_2-plane. Notice that Σ invariant almost by definition: $f(\Sigma) \cap B_1 \subset \Sigma$. The next result shows that Σ is the graph of a C^2 function $g^\star$ from $\Pi(\Sigma)$ to $B_{2\varepsilon_0}$.

Theorem 3.5.1. *Suppose that $f, \ell,$ and Σ satisfy the conditions (3.39)–(3.45) and* *that the following inequalities hold,*

$$\lambda_2 - \epsilon > 0 \ \ , \ \ (\lambda_3 + \epsilon)(\lambda_2 - \epsilon)^{-1} < 1 \tag{3.46}$$

$$(\lambda_3 + \epsilon)(1 + \frac{\epsilon}{\lambda_2 - \epsilon}) < 1 \tag{3.47}$$

$$\frac{\lambda_3 + 2\epsilon}{\lambda_2 - 2\epsilon} < 1 \ \ , \ \ \frac{\lambda_3 + \epsilon}{\lambda_2 - 2\epsilon} + \epsilon\frac{\lambda_3 + 2\epsilon}{(\lambda_2 - 2\epsilon)^2} < 1 \tag{3.48}$$

$$\mu = \frac{\lambda_3 + \epsilon}{(\lambda_2 - 2\epsilon)^2} + \frac{\epsilon(\lambda_3 + 2\epsilon)}{(\lambda_2 - 2\epsilon)^3} < 1. \tag{3.49}$$

Then, there is a C^2 function $g^\star : \Pi(\Sigma) \to B_{2\varepsilon_0}$, such that $g^\star(0,0) = 0$ and $\Sigma = \{(x, g^\star(x)) | x \in \Pi(\Sigma) \subset B_{1\varepsilon_0}\}$.

Moreover, we have

$$|g^\star| \leq \varepsilon_0 \tag{3.50}$$

$$|Dg^\star| \leq \frac{\epsilon}{\lambda_2 - \lambda_3 - 3\epsilon} \tag{3.51}$$

$$|D^2 g^\star| \leq \frac{4M_1(\lambda_2 + \lambda_3)}{(\lambda_2 - 2\epsilon)^3(1 - \mu)}. \tag{3.52}$$

Once we have obtained $g^\star$ as in the above theorem, we consider the graph map: $h(x_1, x_2) = (x_1, x_2, g^\star(x_1, x_2))$. This is a C^2 diffeomorphism, and we can conjugate f to $\widetilde{f} : R^2 \to R^2$ defined by

$$h^{-1}fh(x_1, x_2) = \widetilde{f}(x_1, x_2).$$

This means, of course, that we can replace f by the two dimensional diffeomorphism $\widetilde{f}$.

We can also replace the flow ϕ_t restricted to Σ to a two-dimensional flow $\widetilde{\phi}_t = h^{-1}\phi_t h$.

The first step in the proof of Theorem 3.5.1 is to show that the upper part D_0 of Σ is a C^2 graph. That is,

Lemma 3.5.2. *The set D_0 is the graph*

$$\{(x_1, x_2, x_3) : x_3 = g_0(x_1, x_2))\}$$

of a C^2 function g_0 defined on $\Pi(D_0)$.

As we will show, this holds because X is C^1 close to its linear part, and the analogous statement is true for the linear part.

Indeed, let us consider the linear vector field $L(x_1, x_2, x_3) = (ax_1, bx_2, cx_3)$.

Let $\psi(t, x) = (\psi_1(t, x), \psi_2(t, x), \psi_3(t, x))$ denote the flow of L, and let $T > 0$ be such that if $x_0 \in \ell$ and $|t| < T$, then

$$|\psi_1(t, x_0)| < \frac{b\varepsilon_0}{c - a}. \tag{3.53}$$

Let

$$\widetilde{D_0} = \{\psi(t,x) : |t| < T, x \in \ell\}.$$

Proposition 3.5.3. *Under the conditions above, the set* $\widetilde{D_0}$ *is the graph of the (real-analytic) function*

$$x_3 = \theta x_1 (\frac{x_2}{\varepsilon_0})^{\frac{c-a}{b}}$$

whose first order partial derivatives are bounded by 1.

Proof: First, let us write down the solution of the linear ODE with initial condition $(x_{10}, x_{20}, x_{30}) \in B_1$ as follows:

$$\begin{aligned} x_1 &= e^{at} x_{10} & (3.54) \\ x_2 &= e^{bt} x_{20} & (3.55) \\ x_3 &= e^{ct} x_{30}. & (3.56) \end{aligned}$$

We want to represent x_3 in terms of x_1 and x_2, i.e., $x_3 = g_0(x_1, x_2)$, the graph of which passes through the initial line segment l. By (3.55), we have

$$e^t = (\frac{x_2}{\varepsilon_0})^{\frac{1}{b}}. \tag{3.57}$$

Plugging it into (3.54), we have

$$x_{10} = x_1 (\frac{x_2}{\varepsilon_0})^{\frac{-a}{b}}.$$

Since $x_{30} = \theta x_{10}$, together with the above equation and (3.57), solution (3.56) becomes

$$x_3 = (\frac{x_2}{\varepsilon_0})^{\frac{c}{b}} \cdot \theta x_1 (\frac{x_2}{\varepsilon_0})^{\frac{-a}{b}} = \theta x_1 (\frac{x_2}{\varepsilon_0})^{\frac{c-a}{b}}.$$

Let $\beta = \frac{c-a}{b}$, which satisfies $\beta > 2$, because of (3.37).

It is straightforward to compute that

$$\frac{\partial x_3}{\partial x_1} = \theta (\frac{x_2}{\varepsilon_0})^{\beta}$$

and

$$\frac{\partial x_3}{\partial x_2} = \frac{\theta \beta x_1}{\varepsilon_0} (\frac{x_2}{\varepsilon_0})^{\beta - 1}.$$

Now, we have $0 \leq x_2 \leq \varepsilon_0$, and $|\theta| \leq 1$. Further, (3.53) gives $|x_1| < \frac{\varepsilon_0}{\beta}$, so we get that $|\frac{\partial x_3}{\partial x_1}|$ and $|\frac{\partial x_3}{\partial x_2}|$ are bounded by 1, and the proposition is proved.

Using Lemma 3.5.5, let us prove the following lemma.

Lemma 3.5.4. *Given such D_0 that contains l in B_1, D_0 can be represented as the graph of a C^2 function $g_0(x_1, x_2)$ for $(x_1, x_2) \in \Pi(D_0)$, where Π is the projection into the $x_1 x_2$-plane. Moreover, we have*

$$Lip(g_0) = \sup_{x \in \Pi(D_0)} |Dg_0| \leq 1$$

and $D^2 g_0(x)$ is uniformly bounded in $\Pi(D_0)$.

Proof:

Let the solution of the associated linear vector field of X be:

$$\begin{aligned}
\psi_1(t, l(0)+sv) &= e^{at}s \\
\psi_2(t, l(0)+sv) &= e^{bt}\varepsilon_0 \\
\psi_3(t, l(0)+sv) &= e^{ct}(w+\theta s).
\end{aligned}$$

We have the solution of the vector field X:

$$\begin{aligned}
x_1(t,s) &= \varphi_1(t, l(0)+sv) = \psi_1 + \xi_1 \\
x_2(t,s) &= \varphi_2(t, l(0)+sv) = \psi_2 + \xi_2 \\
x_3(t,s) &= \varphi_3(t, l(0)+sv) = \psi_3 + \xi_3
\end{aligned}$$

where ψ_i's and ξ_i's are evaluated at $(t, l(0)+sv)$. In order to find $g_0(x_1, x_2)$, we express (t, s) as functions of (x_1, x_2), so that we can write

$$g_0(x_1, x_2) = \varphi_3(t(x_1, x_2), l(0) + s(x_1, x_2)v).$$

This requires that the determinant of the Jacobian does not equal to zero:

$$
\begin{aligned}
DET = \det|\frac{\partial(x_1, x_2)}{\partial(t, s)}| &= \det \begin{pmatrix} x_{1,t} & x_{1,s} \\ x_{2,t} & x_{2,s} \end{pmatrix} \\
&= \det \begin{pmatrix} \psi_{1,t} + \xi_{1,t} & \psi_{1,s} + \xi_{1,s} \\ \psi_{2,t} + \xi_{2,t} & \psi_{2,s} + \xi_{2,s} \end{pmatrix} \\
&= \det \begin{pmatrix} ae^{at}s + \xi_{1,t} & e^{at} + \xi_{1,s} \\ be^{bt}\varepsilon_0 + \xi_{2,t} & 0 + \xi_{2,s} \end{pmatrix} \\
&= (ae^{at}s + \xi_{1,t})\xi_{2,s} - (e^{at} + \xi_{1,s})(be^{bt}\varepsilon_0 + \xi_{2,t})
\end{aligned}
$$

We have

$$DET \approx b\varepsilon_0 e^{(a+b)t} \neq 0 \tag{3.58}$$

when ξ_i's are small enough.

Here we use the notation $x_{i,t} = \frac{\partial x_i}{\partial t}$ and $x_{i,s} = \frac{\partial x_i}{\partial s}$ for $i = 1, 2$.

In addition, since

$$\frac{\partial(t, s)}{\partial(x_1, x_2)} = \begin{pmatrix} t_{x_1} & t_{x_2} \\ s_{x_1} & s_{x_2} \end{pmatrix} = \frac{1}{DET} \begin{pmatrix} x_{2,s} & -x_{1,s} \\ -x_{2,t} & x_{1,t} \end{pmatrix}$$

we have

$$\begin{pmatrix} t_{x_1} & t_{x_2} \\ s_{x_1} & s_{x_2} \end{pmatrix} = \frac{1}{DET} \begin{pmatrix} \xi_{2,s} & -e^{at} - \xi_{1,s} \\ -be^{bt}\varepsilon_0 - \xi_{2,t} & ae^{at}s + \xi_{1,t} \end{pmatrix}.$$

So we may compute the first derivatives of g_0:

$$
\begin{aligned}
|\frac{\partial g_0(x_1,x_2)}{\partial x_1}| &= |\frac{\partial \varphi_3}{\partial t}\cdot\frac{\partial t}{\partial x_1}+\frac{\partial \varphi_3}{\partial s}\cdot\frac{\partial s}{\partial x_1}| \\
&= |\frac{(\psi_{3,t}+\xi_{3,t})\cdot\xi_{2,s}}{DET}+\frac{(\psi_{3,s}+\xi_{3,s})(-be^{bt}\varepsilon_0-\xi_{2,t})}{DET}| \\
&= |\frac{(ce^{ct}(w+\theta s)+\xi_{3,t})\cdot\xi_{2,s}}{DET}+\frac{(e^{ct}\theta+\xi_{3,s})(-be^{bt}\varepsilon_0-\xi_{2,t})}{DET}| \\
&\approx |\frac{-b\theta\varepsilon_0 e^{(b+c)t}}{b\varepsilon_0 e^{(a+b)t}}| \text{ , by (3.58)} \\
&= |\theta e^{(c-a)t}| < 1
\end{aligned}
$$

where $\xi_{i,j}$ for $i=2,3,\ j=t,s$ are small enough.

Similarly, we have

$$
\begin{aligned}
|\frac{\partial g_0(x_1,x_2)}{\partial x_2}| &= |\frac{\partial \varphi_3}{\partial t}\cdot\frac{\partial t}{\partial x_2}+\frac{\partial \varphi_3}{\partial s}\cdot\frac{\partial s}{\partial x_2}| \\
&= |\frac{(\psi_{3,t}+\xi_{3,t})\cdot(-e^{at}-\xi_{1,s})}{DET}+\frac{(\psi_{3,s}+\xi_{3,s})(ae^{at}s+\xi_{1,t})}{DET}| \\
&= |\frac{(ce^{ct}(w+\theta s)+\xi_{3,t})\cdot(-e^{at}-\xi_{1,s})}{DET} \\
&\quad +\frac{(e^{ct}\theta+\xi_{3,s})(ae^{at}s+\xi_{1,t})}{DET}| \\
&\approx |\frac{-ce^{(a+c)t}(w+\theta s)+a\cdot s\theta e^{(a+c)t}}{b\varepsilon_0 e^{(a+b)t}}| \text{ , by (3.58)} \\
&\approx |\frac{\theta(a-c)s\cdot e^{(a+c)t}}{b\varepsilon_0 e^{(a+b)t}}| \text{ , where } w \text{ is close to } 0 \\
&\leq |\theta|\cdot|\frac{(a-c)s}{b\varepsilon_0}|e^{(c-b)t} < 1
\end{aligned}
$$

where $\xi_{i,j}$ for $i = 1, 3$, $j = t, s$ are small and $|\frac{(a-c)s}{b\varepsilon_0}| < 1$.

Since g_0 is C^2, all its second derivatives are uniformly bounded over $\Pi(D_0) \subset B_{1\varepsilon_0}$, the lemma is proved.

Lemma 3.5.5. *Let $\phi_t(x)$ be the solution of the vector field*

$$X(x) = (ax_1 + \bar{X}_1(x), bx_2 + \bar{X}_2(x), cx_3 + \bar{X}_3(x)) = (L + \bar{X})(x),$$

where $a > 0 > b > c$ and $-c > a$, with

$$|\bar{X}_i| \leq \epsilon_1, \ |\bar{X}_{i,x_j}| \leq \epsilon_2, \ |D^2\bar{X}| \leq M_1, \ \textit{for } i, j = 1, 2, 3.$$

Let $\psi_t(x)$ be the solution of the corresponding linear vector field, then in B_1, there exists small constants δ_0, δ and a constant M_2, such that,

$$|\phi_t(x) - \psi_t(x)| \leq \delta_0,$$

$$|D\phi_t(x) - D\psi_t(x)| \leq \delta,$$

and

$$|D^2\phi_t(x) - D^2\psi_t(x)| = |D^2\phi_t(x)| \leq M_2.$$

Proof: By definition, we have

$$\begin{aligned}\phi_t(x) - \psi_t(x) &= x + \int_0^t (L + \bar{X})(\phi_s(x))ds - x - \int_0^t L\psi_s(x)ds \\ &= \int_0^t \bar{X}(\phi_s(x))ds + \int_0^t L(\phi_s - \psi_s)(x)ds.\end{aligned}$$

So by the Gronwall's inequality, we have

$$\begin{aligned}|\phi_t(x) - \psi_t(x)| &\leq \int_0^t |\bar{X}(\phi_s(x))|ds + \int_0^t |L||\phi_s(x) - \psi_s(x)|ds \\ &\leq \epsilon_1 t e^{|c|t} \leq \epsilon_1 T e^{|c|T} = \delta_0\end{aligned}$$

Next, let us compare $D\phi_t(x)$ with $D\psi_t(x)$, which come from the first variational equations of $\phi_t(x)$ and $\psi_t(x)$. We have

$$D\dot{\phi}_t(x) = DX_{\phi_t(x)} \cdot D\phi_t(x) = (DL + D\bar{X}_{\phi_t(x)})D\phi_t(x).$$

so

$$|D\dot{\phi}_t(x)| \leq |Id| + \int_{s=0}^t |DL + D\bar{X}_{\phi_s(x)}||D\phi_s(x)|ds.$$

By our assumption in this lemma, we have each entry $\bar{X}_{i,x_j}(\phi_s(x))$, $i, j = 1, 2, 3$, of the 3×3 matrix $D\bar{X}_{\phi_s(x)}$ is bounded by ϵ_2, thus

$$|D\bar{X}_{\phi_s(x)}| = \max_{i=1,2,3}\{|\bar{X}_{i,x_1}| + |\bar{X}_{i,x_2}| + |\bar{X}_{i,x_3}|\} \leq 3\epsilon_2.$$

Then by the Gronwall's inequality we have

$$|D\phi_t(x)| \leq \exp(|DL + D\bar{X}|t) \leq \exp((|c| + 3\epsilon_2)t). \tag{3.59}$$

Now

$$\begin{aligned}
& D\phi_t(x) - D\psi_t(x) \\
= \; & Id + \int_0^t (DL + D\bar{X}_{\phi_s(x)}) D\phi_s(x) ds - Id - \int_0^t DL \cdot D\psi_s(x) ds \\
= \; & \int_0^t D\bar{X}_{\phi_s(x)} D\phi_s(x) ds + \int_0^t DL(D\phi_s(x) - D\psi_s(x)) ds.
\end{aligned}$$

By the Gronwall's inequality again and the inequality (3.59), we have

$$\begin{aligned}
|D\phi_t(x) - D\psi_t(x)| &\leq 3\epsilon_2 t \exp(|DL + D\bar{X}|t) \exp(|DL|t) \\
&\leq 3\epsilon_2 t \exp((2|DL| + |D\bar{X}|)T) \\
&\leq 3\epsilon_2 T \exp((2|c| + 3\epsilon_2)T) \\
&= \delta.
\end{aligned}$$

Finally, let us find the size of $|D^2\phi_t(x) - D^2\psi_t(x)| = |D^2\phi_t(x) - 0|$. We have

$$\begin{aligned}
\frac{\partial^2 \phi}{\partial x_i \partial x_j}(t, x) &= \frac{\partial}{\partial x_j}(e_i + \int_0^t DX_{\phi_s(x)} \frac{\partial \phi(s, x)}{\partial x_i} ds) \\
&= \int_0^t D^2 X_{\phi_s(x)} \cdot (\frac{\partial \phi}{\partial x_i})(\frac{\partial \phi}{\partial x_j}) + DX_{\phi_s(x)} \cdot \frac{\partial^2 \phi(s, x)}{\partial x_i \partial x_j} ds.
\end{aligned}$$

By using (3.59) and the Gronwall's inequality again, and since $D^2X_{\phi_s(x)} = D^2\bar{X}_{\phi_s(x)}$

we have

$$
\begin{aligned}
|\frac{\partial^2\phi}{\partial x_i \partial x_j}(t,x)| &\le \int_0^t M_1 \exp(2s(|c|+3\epsilon_2))ds + \int_0^t (|c|+3\epsilon_2)\cdot|\frac{\partial^2\phi(s,x)}{\partial x_i \partial x_j}|ds \\
&\le \frac{M_1}{2(|c|+3\epsilon_2)}[\exp(2T(|c|+3\epsilon_2)) - 1]\cdot \exp(T(|c|+3\epsilon_2)) \\
&= M_2.
\end{aligned}
$$

Thus the lemma is proved.

Let us be briefly reminded of the definition of our surface Σ. We have $D_0 = \bigcup_{0\le t\le T}\varphi_t(l) \subset int(B_1)$ for our initial line segment l. By now we have shown that D_0 can be represented as a graph of a C^2 function g_0. Now let D_1 be the connected component of $f(D_0)\cap B_1$. And inductively, let D_n be the connected component of $f(D_{n-1})\cap B_1$ which is connected to D_n. We define $\Sigma = \bigcup_{j=0}^{\infty} D_j$.

Theorem 3.5.6. *Σ can be represented as the graph of a C^2 function $g^\star$.*

We will use the same Fiber Contraction method as in the previous section. The proof will require several steps. It involves obtaining a sequence g_1, g_2, $\cdots$, of C^2 functions, such that

$$g_n \text{ converges to } g^\star \text{ uniformly on } B_1, \tag{3.60}$$

$$Dg_n \text{ converges uniformly on } B_1, \tag{3.61}$$

$$D^2 g_n \text{ converges uniformly on } B_1. \tag{3.62}$$

The proof involves definitions of certain function spaces and associated mappings. Let us proceed to define our first function space $\mathcal{G}$. Here we recall that $B_{1\varepsilon_0}$ is $B_1 \cap x_1x_2$-plane, and $B_{2\varepsilon_0}$ is $B_1 \cap x_3$-axis. Let

$$\mathcal{B}_1^0 = \{g | g \in \mathcal{C}^0(\Pi(\Sigma), B_{2\varepsilon_0}), Lip(g) = \sup_{\xi_1 \neq \xi_2} \frac{|g(\xi_1) - g(\xi_2)|}{|\xi_1 - \xi_2|} \leq 1\}.$$

$\mathcal{B}_1^0$ is a complete metric space with respect to the supremum norm (defined above). The subspace

$$\mathcal{G} = \{g |\ g \in \mathcal{B}_1^0,\ g(x_1, 0) = 0 \ for\ |x_1| \leq \varepsilon_0,\ g|_{\Pi(D_0)} = g_0\}$$

is also a complete metric space equipped with the metric

$$d(g_1, g_2) = \sup_{x \in B_{1\varepsilon_0}} |g_1(x) - g_2(x)|.$$

For $x \in \Pi(\Sigma) \cap \Pi(f(\Sigma))$, we have the following graph transform formula

$$\Gamma_f(g) = f_2 \circ (1, g) \circ [f_1 \circ (1, g)]^{-1}.$$

We need to show that $\Gamma_f : \mathcal{G} \to \mathcal{G}$ is a well defined contraction map. This will give us a continuous function g as the unique attractive fixed point of Γ.

Lemma 3.5.7. *Given f satisfying the conditions at the beginning of this section, if*

$\lambda_2 - \epsilon > 0$ and $(\lambda_3 + \epsilon)(\lambda_2 - \epsilon)^{-1} \leq 1$ in B_1, then $\Gamma_f(g)$ is well-defined and maps $\mathcal{G}$ to itself. If, in addition,

$$\eta = (\lambda_3 + \epsilon)(1 + \frac{\epsilon}{\lambda_2 - \epsilon}) < 1$$

then $\Gamma_f(g)$ is a contraction mapping with contraction μ. The unique fixed point of $\Gamma_f(g)$ is a C^0 function $g^\star$ with

$$|g^\star| \leq \frac{\eta}{1-\eta}|g_1 - g_0|.$$

Proof: To show that Γ_f is well-defined, it is sufficient to show that $f_1 \circ (1, g)$ is invertible. Let E denote the x_1x_2-plane. For $u = (u_1, u_2) \in E$, $(L \mid E)(u) = (\lambda_1 u_1, \lambda_2 u_2)$.

Let

$$min(L \mid E) = \inf_{|u|=1} |(L \mid E)u| > \lambda_2 > 0.$$

For two points $u \neq v \in B_{1\varepsilon_0}$, we have

$$
\begin{aligned}
|f_1 \circ (1,g)(u) - f_1 \circ (1,g)(v)| &= |(L \mid E)(u-v) + \tilde{f}_1(u,gu) - \tilde{f}_1(v,gv)| \\
&\geq min(L \mid E)|u-v| - |\tilde{f}_1(u,gu) - \tilde{f}_1(v,gv)| \\
&\geq \lambda_2|u-v| - Lip(\tilde{f}_1)Lip(1,g)|u-v|.
\end{aligned}
$$

Since $Lip(\tilde{f}_1) \leq \epsilon$ and $Lip(1,g) \leq 1$, we have

$$
|f_1 \circ (1,g)(u) - f_1 \circ (1,g)(v)| \geq (\lambda_2 - \epsilon)|u-v| > 0. \tag{3.63}
$$

Hence, on $B_{1\varepsilon_0}$, $f_1 \circ (1,g)$ is injective, and thus, invertible. That also implies

$$
Lip([f_1 \circ (1,g)]^{-1}) \leq (\lambda_2 - \epsilon)^{-1}.
$$

In order to have $\Gamma_f(g)$ map $\mathcal{G}$ to itself, we first prove that for $x_1 \in [0, \varepsilon_0]$ we have

$$
\Gamma_f(g)(x_1, 0) = 0.
$$

Let us be reminded that, as a part of our assumptions on f, we have $W^u_{loc}(\mathbf{0})$ of f is in the x_1-axis inside B_1. In addition, we have had $\Gamma_f(g) = f_2 \circ (1,g) \circ [f_1 \circ (1,g)]^{-1}$ being well defined, i.e., $f_1 \circ (1,g)$ is a one-to-one invertible map. So we know there is a unique $(\bar{x}_1, 0) \in B_{1\varepsilon_0}$ with $|\bar{x}_1| \leq |x_1|$ such that

$$
f_1 \circ (1,g)(\bar{x}_1, 0) = (x_1, 0)
$$

where $|\bar{x}_1| = |x_1|$ is achieved when $x_1 = 0$. So

$$\Gamma_f(g)(x_1, 0) = f_2 \circ (1, g)(\bar{x}_1, 0) = f_2(\bar{x}_1, 0, g(\bar{x}_1, 0)).$$

For $g \in \mathcal{G}$, we have $g(\bar{x}_1, 0) = 0$, so

$$\Gamma_f(g)(x_1, 0) = f_2(\bar{x}_1, 0, 0) = 0.$$

Next, let us prove $Lip(\Gamma_f(g)) \leq 1$ as follows.

Again, for $u = (u_1, u_2), v = (v_1, v_2) \in B_{1\varepsilon_0}$, we have

$$|f_2 \circ (1, g)(u) - f_2 \circ (1, g)(v)| = |\lambda_3 g(u) - \lambda_3 g(v) + \tilde{f}_2(u, gu) - \tilde{f}_2(u, gv)|. \quad (3.64)$$

Since $Lip(\tilde{f}_2) \leq \epsilon$, $Lip(1, g) \leq 1$, we have

$$|f_2 \circ (1, g)(u) - f_2 \circ (1, g)(v)| \leq (\lambda_3 + \epsilon)|u - v|. \quad (3.65)$$

As we know the Lipschitz constant of the composite function of any f and g over domain U can be computed as

$$\begin{aligned} Lip(f \circ g(x)) &= \sup_{x,y \in U} \frac{|f(g(x)) - f(g(y))|}{|x - y|} \\ &\leq \sup_{x,y \in U} \frac{Lip(f)|g(x) - g(y)|}{|x - y|} = Lip(f)Lip(g) \end{aligned}$$

we have

$$Lip(\Gamma_f(g)) \leq (\lambda_3 + \epsilon)(\lambda_2 - \epsilon)^{-1}.$$

Under the condition that $(\lambda_3 + \epsilon)(\lambda_2 - \epsilon)^{-1} \leq 1$, we have $Lip(\Gamma_f(g)) \leq 1$, and thus, $\Gamma_f(g)$ maps $\mathcal{G}$ to itself.

Now let us show Γ is a contraction mapping in $\mathcal{G}$. Let $\xi_1 = [f_1 \circ (1, g_1)]^{-1}$, $\xi_2 = [f_1 \circ (1, g_2)]^{-1}$, we have

$$\begin{aligned}
& |\Gamma_f(g_1) - \Gamma_f(g_2)| \qquad (3.66)\\
= \; & |f_2 \circ (1, g_1) \circ [f_1 \circ (1, g_1)]^{-1} - f_2 \circ (1, g_2) \circ [f_1 \circ (1, g_2)]^{-1}| \\
= \; & |f_2 \circ (1, g_1)\xi_1 - f_2 \circ (1, g_2)\xi_2)| \\
\leq \; & |f_2 \circ (1, g_1)\xi_1 - f_2 \circ (1, g_2)\xi_1| + |f_2 \circ (1, g_2)\xi_1 - f_2 \circ (1, g_2)\xi_2)| \\
\leq \; & Lip(f_2)|(\xi_1, g_1\xi_1) - (\xi_1, g_2\xi_1)| + Lip(f_2)Lip((1, g_2))|\xi_1 - \xi_2| \\
\leq \; & Lip(f_2)|g_1\xi_1 - g_2\xi_1| + Lip(f_2)|\xi_1 - \xi_2| \\
= \; & Lip(f_2)|g_1 - g_2| + Lip(f_2)|\xi_1 - \xi_2|.
\end{aligned}$$

So we need to find $|\xi_1 - \xi_2|$. Claim:

$$|\xi_1 - \xi_2| \leq \frac{\epsilon}{\lambda_2 - \epsilon}|g_1 - g_2|. \qquad (3.67)$$

Let us prove this claim. In our notations, $\xi_1 = [f_1 \circ (1, g_1)]^{-1}$, $\xi_2 = [f_1 \circ (1, g_2)]^{-1}$, let $\tau_1 = \xi_1 x$ and $\tau_2 = \xi_2 x$, then we have $x = \xi_1^{-1}\tau_1 = f_1 \circ (1, g_1)\tau_1$, $x = \xi_2^{-1}\tau_2 =$

$f_1 \circ (1, g_2)\tau_2$. So

$$f_1 \circ (1, g_1)\tau_1 = f_1 \circ (1, g_2)\tau_2.$$

As before, we write

$$f_1(\tau_1, g_1\tau_1) = L^c\tau_1 + \tilde{f}_1(\tau_1, g_1\tau_1),$$

and

$$f_1(\tau_2, g_2\tau_2) = L^c\tau_2 + \tilde{f}_1(\tau_2, g_2\tau_2).$$

So we have

$$\begin{aligned} L^c\tau_1 + \tilde{f}_1(\tau_1, g_1\tau_1) &= L^c\tau_2 + \tilde{f}_1(\tau_2, g_2\tau_2) \\ L^c(\tau_2 - \tau_1) &= \tilde{f}_1(\tau_1, g_1\tau_1) - \tilde{f}_1(\tau_2, g_2\tau_2) \end{aligned}$$

then,

$$L^c(\tau_2 - \tau_1) = \tilde{f}_1(\tau_1, g_1\tau_1) - \tilde{f}_1(\tau_2, g_1\tau_2) + \tilde{f}_1(\tau_2, g_1\tau_2) - \tilde{f}_1(\tau_2, g_2\tau_2)$$

which implies

$$\begin{aligned} L^c(\tau_2 - \tau_1) - \tilde{f}_1(\tau_1, g_1\tau_1) + \tilde{f}_1(\tau_2, g_1\tau_2) &= \tilde{f}_1(\tau_2, g_1\tau_2) - \tilde{f}_1(\tau_2, g_2\tau_2) \\ f_1 \circ (1, g_1)\tau_2 - f_1 \circ (1, g_1)\tau_1 &= \tilde{f}_1(\tau_2, g_1\tau_2) - \tilde{f}_1(\tau_2, g_2\tau_2). \end{aligned}$$

By (3.63), i.e.,

$$|f_1 \circ (1, g_1)\tau_2 - f_1 \circ (1, g_1)\tau_1| \geq (\lambda_2 - \epsilon)|\tau_1 - \tau_2|$$

we have

$$\begin{aligned}
(\lambda_2 - \epsilon)|\tau_1 - \tau_2| &\leq |\tilde{f}_1(\tau_2, g_1\tau_2) - \tilde{f}_1(\tau_2, g_2\tau_2)| \\
&\leq Lip(\tilde{f}_1)|(1, g_1)\tau_2 - (1, g_2)\tau_2| \\
&\leq \epsilon|g_1\tau_2 - g_2\tau_2| \\
&\leq \epsilon|g_1 - g_2|.
\end{aligned}$$

By our notations $\tau_1 = \xi_1 x$ and $\tau_2 = \xi_2 x$, so

$$|\xi_1 - \xi_2| \leq \frac{\epsilon}{\lambda_2 - \epsilon}|g_1 - g_2|$$

and the claim (3.67) is proved. Plugging (3.67) to (3.66), we have

$$|\Gamma_f(g_1) - \Gamma_f(g_2)| \leq Lip(f_2)|g_1 - g_2| + Lip(f_2)\frac{\epsilon}{\lambda_2 - \epsilon}|g_1 - g_2|.$$

As we write $f_2(x_1, x_2) = L^{ss}x_2 + \tilde{f}_2(x_1, x_2)$ with $Lip(\tilde{f}_2) \leq \epsilon$, we have

$$Lip(f_2) \leq \lambda_3 + \epsilon.$$

Hence,

$$|\Gamma_f(g_1) - \Gamma_f(g_2)| \leq (\lambda_3 + \epsilon)(1 + \frac{\epsilon}{\lambda_2 - \epsilon})|g_1 - g_2|.$$

We may observe that $(\lambda_3 + \epsilon)(1 + \frac{\epsilon}{\lambda_2 - \epsilon})$ is approximately equal to λ_3, which is much less than 1. If the neighborhood is properly chosen, we will have

$$\eta = (\lambda_3 + \epsilon)(1 + \frac{\epsilon}{\lambda_2 - \epsilon}) < 1.$$

So the Lemma is proved.

Thus, there is a unique globally attracting fixed point $g^\star \in \mathcal{G}$ for the operator Γ.

To show that $g^\star$ is, in fact, C^2, and to estimate its C^2 size, we modify the fiber contraction procedure of Hirsch and Pugh familiar from invariant manifold theory [4].

We construct two product bundles $\mathcal{G} \times \mathcal{H}$ and $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$ and maps

$$\Phi : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H},$$

$$\Psi : \mathcal{G} \times \mathcal{H} \times \mathcal{K} \to \mathcal{G} \times \mathcal{H} \times \mathcal{K}$$

satisfying the following properties:

1. Φ is a fiber contraction over Γ,

2. Ψ is a fiber contraction over Φ,

3. if g is any map in $\mathcal{G}$ which happens to be C^2 and (g, Dg, D^2g) is in $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$,

then

(a) $\Phi(g, Dg) = (\Gamma(g), D\Gamma(g))$, and

(b) $\Psi(g, Dg, D^2 g) = (\Gamma(g), D\Gamma(g), D^2\Gamma(g))$.

It will follow that, if $\bar{g} \in \mathcal{G}$ is C^2, and we set $g_n = \Gamma^n(\bar{g})$ for each $n > 0$, then the convergence conditions (3.60), (3.61), and (3.62) will hold, and the proof that $g^\star$ is C^2 will be complete.

In the process, we will also obtain bounds for the first and second derivatives of $g^\star$.

We have several tasks ahead.

First, we find a C^2 initial function $\bar{g}$ in $\mathcal{G}$.

Recall that our original function g_0 on $\Pi(D_0)$ was C^2.

Now, choose a smooth bump function $\rho : R^2 \to R$ such that $\rho(x) = 1$ on a neighborhood of $\Pi(D_0)$ and $\rho(x) = 0$ on a neighborhood of $B_{1bot} = \{(x_1, 0, 0) | 0 \leq x_1 \leq \varepsilon_0\}$. Then, letting $\nu(x)$ be the identically 0 function on R^2, and define $\bar{g}$ by

$$\bar{g} = \rho(x) g_0 + (1 - \rho(x))\nu(x).$$

The function $\bar{g}$ is clearly C^2 and is evidently in $\mathcal{G}$.

Next, we proceed to define the function space $\mathcal{H}$ and the map Φ.

Let $\mathcal{H} = C^0_1(\Pi(\Sigma), L(R^2, R))$ be the set of continuous functions H from $\Pi(\Sigma)$ to the linear maps from R^2 to R, and $|H| \leq 1$. $\mathcal{H}$ becomes a complete metric space equipped with the metric $d(H_1, H_2) = |H_1 - H_2|$ induced by the norm

$$|H| = \sup_{x \in B_{1\varepsilon_0}, y \in R^2, |y|=1} |H_x(y)|.$$

We can see that $\mathcal{H}$ is the set of candidates of Dg. To get the fiber contraction operator over the functional space $\mathcal{G} \times \mathcal{H}$, we proceed as follows. Let $u(x) = [f_1 \circ (1,g)]^{-1}(x)$. By the chain rule, we take the derivative of $\Gamma_f(g) = f_2 \circ (1,g) \circ [f_1 \circ (1,g)]^{-1}$ with respect to x, we have

$$D_x \Gamma_f(g) = (f_{2x} + f_{2y} D_{u(x)} g)(f_{1x} + f_{1y} D_{u(x)} g)^{-1}$$

where the partial derivatives of f are evaluated at the point $(u(x), g(u(x)))$. In this form, we define

$$R(g, H) = (f_{2x} + f_{2y} H_{u(x)})(f_{1x} + f_{1y} H_{u(x)})^{-1}$$

where the partial derivatives of f are evaluated at the point $(u(x), g(u(x)))$. And let $\Phi : \mathcal{G} \times \mathcal{H} \to \mathcal{G} \times \mathcal{H}$ be

$$\Phi(g, H) = (\Gamma_f(g), R(g, H)).$$

By construction, if g is C^1, then

$$\Phi(g, Dg) = (\Gamma_f(g), D(\Gamma_f(g))).$$

We shall show that $\Phi(g, H) = (\Gamma_f(g), R(g, H))$ is a fiber contraction on $\mathcal{G} \times \mathcal{H}$ with the maximum metric in this product space.

Lemma 3.5.8. *Given f satisfying the conditions at the beginning of this section, if the following conditions are satisfied in B_1*

$$\frac{\lambda_3 + 2\epsilon}{\lambda_2 - 2\epsilon} < 1 \quad \textit{and} \quad \frac{\lambda_3 + \epsilon}{\lambda_2 - 2\epsilon} + \epsilon \frac{\lambda_3 + 2\epsilon}{(\lambda_2 - 2\epsilon)^2} < 1, \tag{3.68}$$

then the function $g^\star$ in Lemma 3.5.7 *is C^1 with*

$$|Dg^\star| \leq \frac{\epsilon}{\lambda_2 - \lambda_3 - 3\epsilon}. \tag{3.69}$$

Proof: Let us prove this lemma by first showing that $\Phi(g, H)$ maps $\mathcal{G} \times \mathcal{H}$ to itself. Since $\Gamma_f(g) : \mathcal{G} \to \mathcal{G}$, we only need to show $R(g, H) : \mathcal{G} \times \mathcal{H} \to \mathcal{H}$. As we can write

$$f_{1x} + f_{1y} H_{u(x)} = f_{1x}(I + f_{1x}^{-1} f_{1y} H_{u(x)}),$$

for $|H| \leq 1$, we have

$$
\begin{aligned}
|(f_{1x} + f_{1y}H_{u(x)})^{-1}| &= |\{f_{1x}(I + f_{1x}^{-1} f_{1y} H_{u(x)})\}^{-1}| \\
&= |(I + f_{1x}^{-1} f_{1y} H_{u(x)})^{-1} f_{1x}^{-1}| \\
&\leq |(I + f_{1x}^{-1} f_{1y} H_{u(x)})^{-1}||f_{1x}^{-1}| \\
&= \frac{1}{min(I + f_{1x}^{-1} f_{1y} H_{u(x)})}|f_{1x}^{-1}| \\
&\leq \frac{1}{1 - |f_{1x}^{-1} f_{1y} H_{u(x)}|}|f_{1x}^{-1}| \\
&\leq \frac{|f_{1x}^{-1}|}{1 - |f_{1x}^{-1}||f_{1y}|} \\
&= \frac{1}{|f_{1x}^{-1}|^{-1} - |f_{1y}|}.
\end{aligned}
$$

Since $|f_{1y}| \leq \epsilon$ and $|f_{1x}^{-1}|^{-1} = min(f_{1x}) \geq \lambda_2 - \epsilon$, we have

$$|(f_{1x} + f_{1y}H_{u(x)})^{-1}| \leq \frac{1}{\lambda_2 - 2\epsilon}.$$

Moreover,

$$
\begin{aligned}
|R(g, H)| &\leq |f_{2x} + f_{2y}H_{u(x)}||(f_{1x} + f_{1y}H_{u(x)})^{-1}| \qquad (3.70) \\
&\leq (\epsilon + (\lambda_3 + \epsilon)|H_{u(x)}|)\frac{1}{\lambda_2 - 2\epsilon} \\
&\leq \frac{\lambda_3 + 2\epsilon}{\lambda_2 - 2\epsilon}.
\end{aligned}
$$

If we have $\frac{\lambda_3+2\epsilon}{\lambda_2-2\epsilon} < 1$ in some neighborhood of 0, we have proved that $\Phi(g, H)$ maps $\mathcal{G} \times \mathcal{H}$ to itself.

Now we shall show $\Phi(g, H)$ is a fiber contraction map. Let $R_2(g, H)_x(y) = (f_{2x} + f_{2y}H_{u(x)})(y)$, $R_1(g, H)_x(y) = ((f_{1x} + f_{1y}H_{u(x)})^{-1})(y)$, both of which are linear operators in y. So we may write $R(g, H)_x = R_2(g, H) \circ R_1(g, H)_x$, where the partial derivatives of f are evaluated at the point $(u(x), g(u(x)))$. By what we have just computed, we know

$$|R_1(g, H)| = |(f_{1x} + f_{1y}H_{u(x)})^{-1}| \leq \frac{1}{\lambda_2 - 2\epsilon} \tag{3.71}$$

and

$$|R_2(g, H)| = |f_{2x} + f_{2y}H_{u(x)}| \leq \lambda_3 + 2\epsilon. \tag{3.72}$$

And by the linearity of R_1 and R_2, we have

$$\begin{aligned}
& |R(g, H_1)x - R(g, H_2)x| \\
\leq\ & |R_2(g, H_1) \circ R_1(g, H_1)x - R_2(g, H_2) \circ R_1(g, H_1)x| \\
& +|R_2(g, H_2) \circ R_1(g, H_1)x - R_2(g, H_2) \circ R_1(g, H_2)x| \\
\leq\ & |R_2(g, H_1) - R_2(g, H_2)||R_1(g, H_1)x| \\
& +|R_2(g, H_2) \circ (R_1(g, H_1)x - R_1(g, H_2)x)| \\
\leq\ & |f_{2y}H_2 - f_{2y}H_1|\frac{1}{\lambda_2 - 2\epsilon} \\
& +|R_2(g, H_2)||R_1(g, H_1)x - R_1(g, H_2)x| \\
\leq\ & (\lambda_3 + \epsilon)|H_2 - H_1|\frac{1}{\lambda_2 - 2\epsilon} \\
& +(\lambda_3 + 2\epsilon)|R_1(g, H_1)x - R_1(g, H_2)x|.
\end{aligned}$$

Here $|R_1(g,H_1)x - R_1(g,H_2)x| = |(f_{1x}+f_{1y}H_1)^{-1} - (f_{1x}+f_{1y}H_2)^{-1}|$ which involves inverse functions. So let us consider this trick:

$$F_1^{-1} - F_2^{-1} = F_1^{-1}(F_2 - F_1)F_2^{-1}$$

which implies

$$|F_1^{-1} - F_2^{-1}| \leq |F_1^{-1}||F_2^{-1}||F_1 - F_2|$$

By the above inequality, we have

$$\begin{aligned}
& |R_1(g,H_1)x - R_1(g,H_2)x| \\
= \; & |(f_{1x}+f_{1y}H_1)^{-1} - (f_{1x}+f_{1y}H_2)^{-1}| \\
\leq \; & |(f_{1x}+f_{1y}H_1)^{-1}||(f_{1x}+f_{1y}H_2)^{-1}||(f_{1x}+f_{1y}H_1) - (f_{1x}+f_{1y}H_2)| \\
\leq \; & (\frac{1}{\lambda_2 - 2\epsilon})^2 |f_{1y}H_1 - f_{1y}H_2| \\
\leq \; & (\frac{1}{\lambda_2 - 2\epsilon})^2 \epsilon |H_1 - H_2|.
\end{aligned}$$

Therefore,

$$|R(g,H_1) - R(g,H_2)| \leq (\frac{\lambda_3+\epsilon}{\lambda_2 - 2\epsilon} + \epsilon\frac{\lambda_3 + 2\epsilon}{(\lambda_2 - 2\epsilon)^2})|H_1 - H_2|.$$

By (3.68), we have

$$\mu = \frac{\lambda_3+\epsilon}{\lambda_2 - 2\epsilon} + \epsilon\frac{\lambda_3 + 2\epsilon}{(\lambda_2 - 2\epsilon)^2} < 1.$$

Hence, Φ is indeed a fiber contraction, and the lemma is proved.

Moreover, we can find the size of $Dg^\star$. By a similar argument in the previous section, we start from (3.70), which says

$$|Dg_n| = |R(g_{n-1}, Dg_{n-1})| \leq (\frac{\epsilon}{\lambda_2 - 2\epsilon} + \frac{\lambda_3 + \epsilon}{\lambda_2 - 2\epsilon}|Dg_{n-1}|)$$

Let

$$J = \frac{\epsilon}{\lambda_2 - 2\epsilon}, \quad and \quad N = \frac{\lambda_3 + \epsilon}{\lambda_2 - 2\epsilon},$$

we have

$$|Dg_n| \leq J + N|Dg_{n-1}|.$$

Similar to the estimate in the inequality (3.30), we have

$$|Dg^\star| \leq \frac{J}{1-N} = \frac{\epsilon}{\lambda_2 - \lambda_3 - 3\epsilon}.$$

In order to have $g^\star$ in C^2, we need another fiber contraction to compute $D^2 g^\star$. We still use the same notation and definition of $\mathcal{G}$ and $\mathcal{H}$. Recall that $\mathcal{H} = C_1^0(\Pi(\Sigma), L(R^2, R))$, now we define another functional space $\mathcal{K}$ to be the set of candidates of $D^2 g$:

$$\mathcal{K} = C^0(\Pi(\Sigma) \times (R^2)^2, R)$$

which is the set of bounded continuous functions K from $\Pi(\Sigma)$ to the bilinear maps from R^2 to R. $\mathcal{K}$ is a complete metric space equipped with the metric $d(K_1, K_2) =$

$|K_1 - K_2|$ induced by the norm

$$|K| = \sup_{x \in B_{1\varepsilon_0}, |v_1|=1, |v_2|=1, v_1, v_2 \in R^2} |K_x(v_1, v_2)|.$$

Similar to the previous fiber contraction constructions, we first use the chain rule to find $D_x^2 \Gamma_f(g)$ which gives the form of our bundle map. As we know

$$\Gamma_f(g) = f_2 \circ (1, g) \circ [f_1 \circ (1, g)]^{-1}.$$

Let $u(x) = [f_1 \circ (1, g)]^{-1}$, we have

$$D_x \Gamma_f(g) = f_{2x}(u(x), g(u(x))) Du(x) + f_{2y}(u(x), g(u(x))) Dg(u(x)) Du(x).$$

Now we take the derivative of the above with respect to x one more time to obtain the following formula, all the derivatives of f are evaluated at $(u(x), g(u(x)))$:

$$\begin{aligned} D_x^2 \Gamma_f(g) &= f_{2xx}(Du(x))^2 + f_{2xy} Dg(u(x))(Du(x))^2 + f_{2x} D^2 u(x) \\ &\quad + \{f_{2yx} Du(x) + f_{2yy} Dg(u(x)) Du(x)\} Dg(u(x)) Du(x) \\ &\quad + f_{2y}\{D^2 g(u(x))(Du(x))^2 + Dg(u(x)) D^2 u(x)\}. \end{aligned} \tag{3.73}$$

Now let us explicitly compute $Du(x)$ and $D^2 u(x)$. We have

$$Du(x) = [f_{1x}(u(x), g(u(x))) + f_{1y}(u(x), g(u(x))) Dg(u(x))]^{-1}$$

and with all the derivatives of f evaluated at $(u(x), g(u(x)))$:

$$\begin{aligned} D^2u(x) &= -Du(x)\{f_{1xx}(Du(x))^2 + 2f_{1xy}Dg(u(x))(Du(x))^2 \\ &\quad + f_{1yy}(Dg(u(x)))^2(Du(x))^2 + f_{1y}(Du(x))^2 D^2g(u(x))\}. \end{aligned}$$

In this form, we define

$$D_1 = D_1(u, H)_x = [f_{1x} + f_{1y}H_{u(x)}]^{-1}$$

and

$$\begin{aligned} D_2 = D_2(u, H, K)_x &= -D_1\{f_{1xx}(D_1)^2 + 2f_{1xy}H_{u(x)}(D_1)^2 \\ &\quad + f_{1yy}(H_{u(x)})^2(D_1)^2 + f_{1y}(D_1)^2 K_{u(x)}\}. \end{aligned}$$

So we define the following functional derived from (3.73)

$$\begin{aligned} \bar{R}(g, H, K) &= f_{2xx}(D_1)^2 + f_{2xy}H_{u(x)}(D_1)^2 + f_{2yx}D_1H_{u(x)}D_1 \\ &\quad + f_{2yy}H_{u(x)}D_1H_{u(x)}D_1 + f_{2x}D_2 \\ &\quad + f_{2y}K_{u(x)}(D_1)^2 + f_{2y}H_{u(x)}D_2 \end{aligned}$$

where the partial derivatives of f are evaluated at the point $(u(x), g(u(x)))$, and only the last three terms contain K. To get the fiber contraction operator in the functional space $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$, we let $\Psi : \mathcal{G} \times \mathcal{H} \times \mathcal{K} \to \mathcal{G} \times \mathcal{H} \times \mathcal{K}$ satisfy this form

$$\Psi(g, Dg, D^2g) = (\Gamma_f(g), D\Gamma_f(g), D^2\Gamma_f(g)).$$

So we write

$$\Psi(g, H, K) = (\Gamma_f(g), R(g, H), \bar{R}(g, H, K))$$

and we shall show that $\Psi(g, H, K)$ is a fiber contraction on $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$ with the maximum metric in this product space.

Lemma 3.5.9. *Given f satisfying the conditions at the beginning of this section. In addition, each one of the second derivatives of f is bounded by some constant $M_1 > 0$. If in B_1 the following condition is satisfied*

$$\mu = \frac{\lambda_3 + \epsilon}{(\lambda_2 - 2\epsilon)^2} + \frac{\epsilon(\lambda_3 + 2\epsilon)}{(\lambda_2 - 2\epsilon)^3} < 1, \tag{3.74}$$

then $g^\star$ is C^2 with

$$|D^2 g^\star| \leq \frac{4M_1(\lambda_2 + \lambda_3)}{(\lambda_2 - 2\epsilon)^3(1 - \mu)}. \tag{3.75}$$

Proof: Let us first prove that Ψ maps $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$ to itself, and it is clear that we only need to show, for given g and H, $\bar{R}(g, H, K)$ is still in $\mathcal{K}$. By our definition

$$D_1 = [f_{1x} + f_{1y} H_{u(x)}]^{-1}$$

and what we have obtained in inequalities (3.71), we have

$$|D_1| = |(f_{1x} + f_{1y} H_{u(x)})^{-1}| \leq \frac{1}{\lambda_2 - 2\epsilon}.$$

Also we have

$$
\begin{aligned}
|D_2| &\le |D_1|\{|f_{1xx}||D_1|^2 + 2|f_{1xy}||D_1|^2 \\
&\quad + |f_{1yy}||D_1|^2 + |f_{1y}||D_1|^2|K|\} \\
&= |D_1|^3(|f_{1xx}| + 2|f_{1xy}| + |f_{1yy}| + |f_{1y}||K|) \\
&\le \frac{4M_1 + \epsilon|K|}{(\lambda_2 - 2\epsilon)^3}.
\end{aligned}
$$

In addition, since $|H| \le 1$ and each one of the second derivatives of f is bounded by some constant M_1, we have

$$
\begin{aligned}
|\bar{R}(g,H,K)| &\le |f_{2xx}||D_1|^2 + 2|f_{2xy}||D_1|^2 + |f_{2yy}||D_1|^2 \\
&\quad + (|f_{2x}| + |f_{2y}|)|D_2| + |f_{2y}||K||D_1|^2 \\
&\le \frac{4M_1}{(\lambda_2 - 2\epsilon)^2} + (\lambda_3 + 2\epsilon)\frac{4M_1 + \epsilon|K|}{(\lambda_2 - 2\epsilon)^3} + \frac{(\lambda_3 + \epsilon)|K|}{(\lambda_2 - 2\epsilon)^2} \\
&= \frac{4M_1}{(\lambda_2 - 2\epsilon)^2} + \frac{4M_1(\lambda_3 + 2\epsilon)}{(\lambda_2 - 2\epsilon)^3} \\
&\quad + \{\frac{\epsilon(\lambda_3 + 2\epsilon)}{(\lambda_2 - 2\epsilon)^3} + \frac{(\lambda_3 + \epsilon)}{(\lambda_2 - 2\epsilon)^2}\}|K|.
\end{aligned} \tag{3.76}
$$

Since K is also bounded, we know $\bar{R}(g,H,K)$ is bounded. Hence, we have proved Ψ maps $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$ to itself. Now let us show the bundle map Ψ is a fiber contraction on $\mathcal{G} \times \mathcal{H} \times \mathcal{K}$. By definition, what we need to show is the map $K \to \bar{R}(g,H,K)$ is a contraction mapping. Again, from

$$
\begin{aligned}
\bar{R}(g,H,K) &= f_{2xx}(D_1)^2 + f_{2xy}H_{u(x)}(D_1)^2 + f_{2yx}D_1H_{u(x)}D_1 \\
&\quad + f_{2yy}H_{u(x)}D_1H_{u(x)}D_1 + f_{2y}K_{u(x)}(D_1)^2 \\
&\quad + (f_{2x} + f_{2y}H_{u(x)})D_2
\end{aligned}
$$

where

$$
\begin{aligned}
D_2 &= -D_1\{f_{1xx}(D_1)^2 + 2f_{1xy}H_{u(x)}(D_1)^2 + f_{1yy}(H_{u(x)})^2(D_1)^2\} \\
&\quad - D_1f_{1y}(D_1)^2K_{u(x)}
\end{aligned}
$$

we notice that only the last two terms of $\bar{R}(g,H,K)$ involve K. We may have the terms that do not contain K canceled if we perform the following substraction, i.e.,

$$
\begin{aligned}
&\bar{R}(g,H,K_1) - \bar{R}(g,H,K_2) \\
= \;& f_{2y}K_1(D_1)^2 - f_{2y}K_2(D_1)^2 \\
&+ (f_{2x} + f_{2y}H)\{-D_1f_{1y}(D_1)^2K_1 + D_1f_{1y}(D_1)^2K_2\}.
\end{aligned}
$$

By using $|H| \leq 1$, we have

$$\begin{aligned}
& |\bar{R}(g,H,K_1) - \bar{R}(g,H,K_2)| \\
\leq \; & |f_{2y}K_1(D_1)^2 - f_{2y}K_2(D_1)^2| \\
& +|f_{2x} + f_{2y}H||D_1 f_{1y}(D_1)^2 K_1 - D_1 f_{1y}(D_1)^2 K_2| \\
\leq \; & (\lambda_3+\epsilon)|K_1(D_1)^2 - K_2(D_1)^2| \\
& (\lambda_3+2\epsilon)|D_1 f_{1y}(D_1)^2||K_1 - K_2| \\
\leq \; & (\lambda_3+\epsilon)|K_1 - K_2||D_1|^2 \\
& (\lambda_3+2\epsilon)\epsilon|D_1|^3|K_1 - K_2| \\
\leq \; & \underbrace{(\frac{\lambda_3+\epsilon}{(\lambda_2-2\epsilon)^2} + \frac{\epsilon(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3})}_{\mu}|K_1 - K_2|.
\end{aligned}$$

We may observe that

$$\mu = \frac{\lambda_3+\epsilon}{(\lambda_2-2\epsilon)^2} + \frac{\epsilon(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3} \approx \frac{\lambda_3}{(\lambda_2)^2}.$$

If close enough to the saddle fixed point and the strong contraction is much smaller than the weak contraction, we may well have $\mu < 1$. So under the condition that

$$\frac{\lambda_3+\epsilon}{(\lambda_2-2\epsilon)^2} + \frac{\epsilon(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3} < 1$$

we have proved that the map $K \to \bar{R}(g,H,K)$ is a contraction mapping. Moreover, we can find the size of $D^2 g^\star$ by essentially the same argument used in the previous section about the iteration (3.70). So we do not repeat the full argument here, the complete details of which has been shown in the previous section. Here we start with the following estimate, derived from (3.76), which says

$$|\bar{R}(g,H,K)| \leq \underbrace{\frac{4M_1}{(\lambda_2-2\epsilon)^2}+\frac{4M_1(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3}}_{J}+\underbrace{\{\frac{\epsilon(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3}+\frac{(\lambda_3+\epsilon)}{(\lambda_2-2\epsilon)^2}\}}_{N}|K|.$$

Let

$$J=\frac{4M_1}{(\lambda_2-2\epsilon)^2}+\frac{4M_1(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3}=\frac{4M_1(\lambda_2+\lambda_3)}{(\lambda_2-2\epsilon)^3}$$

and from the above

$$\mu=\frac{\epsilon(\lambda_3+2\epsilon)}{(\lambda_2-2\epsilon)^3}+\frac{\lambda_3+\epsilon}{(\lambda_2-2\epsilon)^2}<1.$$

Since K's are candidates of $D^2g^\star$, the above inequality gives us a recursive relation as

$$|D^2g_n|=|\bar{R}(g_{n-1},Dg_{n-1},D^2g_{n-1})|\leq J+\mu|D^2g_{n-1}|.$$

Then we have the estimate similar to the one in the inequality (3.30), i.e.,

$$|D^2g^\star|\leq\frac{J}{1-\mu}=\frac{4M_1(\lambda_2+\lambda_3)}{(\lambda_2-2\epsilon)^3(1-\mu)}$$

.

3.6 Straightening Invariant manifolds of a three dimensional map

In this section, we flatten out the local stable and unstable manifolds of a C^2 map $f : R^3 \to R^3$ near its saddle fixed point at $\mathbf{0}$. We will make a C^2 transformation of the neighborhood so that the local invariant manifolds of f are flattened.

First of all, let us state the Hadamard-Perron Theorem (Stable manifold Theorem).

Theorem 3.6.1. *Let $f : M \to M$ be a C^r diffeomorphism of a smooth manifold M, and let p be a hyperbolic fixed point of f with associated splitting $T_pM = E^u_p \oplus E^s_p$. then $W^s(p)$ is a C^r injectively immersed copy of E^s_p and $W^u(p)$ is a C^r injectively immersed copy of E^u_p. Moreover, $W^s(p)$ is tangent at p to E^s_p and $W^u(p)$ is tangent at p to E^u_p.*

Now let us start by assuming $f : R^3 \to R^3$ is C^2 and satisfies the following conditions. Let $L = Df(\mathbf{0}) = (L^u, L^s)$, and $R^2 = E^u \oplus E^s$ be the splitting given by the hyperbolicity of L, where E^u and E^s correspond to the x_1-axis and x_2x_3-plane respectively. So we write

$$f(x, y) = (L^u x + \tilde{f}_1(x, y), L^s y + \tilde{f}_2(x, y)) = (f_1(x, y), f_2(x, y))$$

where $\tilde{f}_1$ and $\tilde{f}_2$ are the higher-order nonlinear terms of f. And let B_{ε_0} be the box around the origin of radius ε_0, $B_{1\varepsilon_0} = B_{\varepsilon_0} \cap x_1$-axis, and $B_{2\varepsilon_0} = B_{\varepsilon_0} \cap x_2x_3$-plane.

And we assume $f = (f_1, f_2, f_3)$ satisfies the following conditions in B_{ε_0}:

1. $L^u = \lambda_1 > 1$, $L^s = (\lambda_2, \lambda_3)$, $0 < \lambda_3 < \lambda_2 < 1$;

2. $\tilde{f}_i(\mathbf{0}) = 0$ and the first partial derivatives $\tilde{f}_{ix_j}(\mathbf{0}) = 0$, for $i, j = 1, 2, 3$;

3. $|\tilde{f}_{ix_j}| \leq \epsilon$ and $|\tilde{f}_{ix_jx_k}| \leq M_2$, for $i, j, k = 1, 2, 3$.

We wish to find a C^2 function $g : B_{1\varepsilon_0} \to B_{2\varepsilon_0}$ whose graph is $W^u_{loc}(\mathbf{0})$ and is invariant by f. In order to find another function whose graph is $W^s_{loc}(\mathbf{0})$, one may set $g : B_{2\varepsilon_0} \to B_{1\varepsilon_0}$. Since the work are essentially the same, we only state the results of g that corresponds to $W^u_{loc}(\mathbf{0})$. By using the same graph transform method appeared in the previous section, it can be shown that there is such a function g and it is C^2. As before, we set

$$\Gamma_f(g) = f_2 \circ (1, g) \circ [f_1 \circ (1, g)]^{-1}$$

and

$$\mathcal{G} = \{g : B_{1\varepsilon_0} \to B_{2\varepsilon_0} |\ g(0) = 0,\ Lip(g) \leq 1\}$$

which is a complete metric space with the metric

$$d(g_1, g_2) = \sup_{x \in B_{1\varepsilon_0}} |g_1(x) - g_2(x)|.$$

We start with $g_0 \equiv 0$, then $g_1 = \Gamma_f(g_0)$, and inductively, $g_n = \Gamma_f(g_{n-1})$, the sequence of which converges to some function $g^\star \in \mathcal{G}$. Since the proof is essentially the same as is shown in the previous section, we do not repeat the work here. We will adapt a few constants, and state the lemmas and theorem accordingly.

As in the previous section,

$$f(x,y) = (L^c x + \tilde{f}_1(x,y), L^{ss} y + \tilde{f}_2(x,y)) = (f_1(x,y), f_2(x,y))$$

with $m(L^c)$ be its weak contraction eigenvalue, and $|L^{ss}|$ be its strong contraction eigenvalue.

In this section, we have

$$f(x,y) = (L^u x + \tilde{f}_1(x,y), L^s y + \tilde{f}_2(x,y)) = (f_1(x,y), f_2(x,y))$$

where

$$m(L^u) = L^u = \lambda_1, \qquad |L^s| = \lambda_2.$$

Hence, we may adapt Lemma 3.5.7 to have

Lemma 3.6.2. *Given f satisfying the conditions at the beginning of this section, if $\lambda_1 - \epsilon > 0$ and $(\lambda_2 + \epsilon)(\lambda_1 - \epsilon)^{-1} \leq 1$ in B_{ε_0}, then $\Gamma_f(g)$ is well-defined and maps $\mathcal{G}$ to itself. If, in addition,*

$$\eta = (\lambda_2 + \epsilon)(1 + \frac{\epsilon}{\lambda_1 - \epsilon}) < 1$$

then $\Gamma_f(g)$ is a contraction mapping with contraction μ. The unique fixed point of

$\Gamma_f(g)$ *is a* C^0 *function* $g^\star$ *with*

$$|g^\star| \leq \frac{\eta}{1-\eta}|g_1 - g_0|.$$

We get the following Lemma based on Lemma 3.5.8.

Lemma 3.6.3. *Given* f *satisfying the conditions at the beginning of this section, if the following conditions are satisfied in* B_{ε_0}

$$\frac{\lambda_2 + 2\epsilon}{\lambda_1 - 2\epsilon} < 1 \quad \textit{and} \quad \frac{\lambda_2 + \epsilon}{\lambda_1 - 2\epsilon} + \epsilon \frac{\lambda_2 + 2\epsilon}{(\lambda_1 - 2\epsilon)^2} < 1$$

then the function $g^\star$ *in Lemma* 3.5.7 *is* C^1 *with*

$$|Dg^\star| \leq \frac{\epsilon}{\lambda_1 - \lambda_2 - 3\epsilon}.$$

Subsequently, we adapt Lemma 3.5.9 into

Lemma 3.6.4. *Given* f *satisfying the conditions at the beginning of this section, in addition, each one of the second derivatives of* f *is bounded by some constant* $M_2 > 0$*. If in* B_{ε_0} *the following condition is satisfied*

$$\mu = \frac{\lambda_2 + \epsilon}{(\lambda_1 - 2\epsilon)^2} + \frac{\epsilon(\lambda_2 + 2\epsilon)}{(\lambda_1 - 2\epsilon)^3} < 1$$

then we have $g^\star$ *is* C^2 *with*

$$|D^2 g^\star| \leq \frac{4M_2(\lambda_1 + \lambda_2)}{(\lambda_1 - 2\epsilon)^3(1-\mu)}$$

.

In summary, analogous to Theorem 3.5.1 we conclude the theorem as follows.

Theorem 3.6.5. *Given that f satisfies the above conditions, if in B_{ε_0} we have the following conditions satisfied*

$$\lambda_1 - \epsilon > 0 \quad , \quad (\lambda_2 + \epsilon)(\lambda_1 - \epsilon)^{-1} \leq 1$$

$$\eta = (\lambda_2 + \epsilon)(1 + \frac{\epsilon}{\lambda_1 - \epsilon}) < 1$$

$$\frac{\lambda_2 + 2\epsilon}{\lambda_1 - 2\epsilon} < 1 \quad , \quad \frac{\lambda_2 + \epsilon}{\lambda_1 - 2\epsilon} + \epsilon \frac{\lambda_2 + 2\epsilon}{(\lambda_1 - 2\epsilon)^2} < 1$$

$$\mu = \frac{\lambda_2 + \epsilon}{(\lambda_1 - 2\epsilon)^2} + \frac{\epsilon(\lambda_2 + 2\epsilon)}{(\lambda_1 - 2\epsilon)^3} < 1$$

then we have a unique C^2 function $g^\star : B_{1\varepsilon_0} \to B_{2\varepsilon_0}$, whose graph is $W^u_{loc}(\mathbf{0})$ of f. Moreover, we have

$$|g^\star| \leq \varepsilon_0 |Dg^\star|$$

$$|Dg^\star| \leq \frac{\epsilon}{\lambda_1 - \lambda_2 - 3\epsilon}$$

$$|D^2 g^\star| \leq \frac{4M_2(\lambda_1 + \lambda_2)}{(\lambda_1 - 2\epsilon)^3(1 - \mu)}$$

.

After we have obtained C^2 parameterizations of both local stable and unstable manifolds of f at $\mathbf{0}$, which are now denoted by g_s, and g_u respectively. Then, in general, $W^s_{loc}(\mathbf{0})$ (respectively $W^u_{loc}(\mathbf{0})$) is not in E^s (respectively E^u). However, we will show there is a C^2 diffeomorphism $\widetilde{f}$ which conjugates to f and has both local stable and unstable manifolds straightened. Let us consider the map

$$\rho : B_{\varepsilon_0} \to V \subset R^3$$

with

$$\rho(x, y) = (x - g_s(y), y - g_u(x)). \tag{3.77}$$

By this definition, ρ is C^2, $\rho(\mathbf{0}) = \mathbf{0}$ as $g_s(0,0) = g_u(0) = 0$, and $D\rho(\mathbf{0})$ is the identity

$$D\rho(\mathbf{0}) = \begin{pmatrix} 1 & -g_s^{'}(0) \\ -g_u^{'}(0) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where we used the properties of the local manifolds that $g_s^{'}(0) = g_u^{'}(0) = 0$. Thus ρ is a diffeomorphism in some neighborhood of $\mathbf{0} \in R^3$ by the inverse function theorem. In $B_{\varepsilon_0} \cap f^{-1}(B_{\varepsilon_0})$, such that $\rho : B_{\varepsilon_0} \cap f^{-1}(B_{\varepsilon_0}) \to V^{'}$, let

$$\widetilde{f} = \rho f \rho^{-1}. \tag{3.78}$$

Then $\widetilde{f} : V^{'} \to V$ is a C^2 diffeomorphism, with $\widetilde{f}(\mathbf{0}) = \mathbf{0}$ and $D\widetilde{f}(\mathbf{0}) = L$. Moreover, in B_{ε_0} the local stable (respectively, unstable) manifold of $\widetilde{f}$ is in E^s, i.e., x_2x_3-plane (respectively E^u, i.e., x_1-axis). We can verify that property as follows:

$$\rho(g_s(y), y) = (g_s(y) - g_s(y), y - g_u(g_s(y))) = (0, y - g_u(g_s(y)))$$

and

$$\rho(x, g_U(x)) = (x - g_S(g_U(x)), g_U(x) - g_U(x)) = (x - g_S(g_U(x)), 0)$$

Thus, up to a smooth conjugacy, the local manifolds of f is straightened.

BIBLIOGRAPHY

BIBLIOGRAPHY

[1] John Guckenheimer and Philip Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcation of Vector Fields*, volume 42 of *Applied Mathematical Sciences*. Springer-Verlap, New York, 1983.

[2] P. Hartman. On local homeomorphisms of euclidean spaces. *Bol. Soc. Mat. Mexicana*, 5:220–241, 1960.

[3] P. Hartman. *Ordinary Differential Equations, 2nd ed.*, volume 38 of *Siam Classics in Applied Math*. SIAM, 2002.

[4] M. Hirsch and C. Pugh. Stable manifolds and hyperbolic sets. *Proc. AMS Symp. Pure Math.*, 14, 1970.

[5] J. Palis and F. Takens. *Hyperbolicity and Sensitive Chaotic Dynamics at Homoclinic Bifurcations.*, volume 35 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 1993.

[6] Clark Robinson. *Dynamical Systems, Stability, Symbolic Dynamics, and Chaos, second ed.* Studies in Advanced Mathematics. CRC Press, 1999.

[7] Colin Sparrow. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*, volume 41 of *Applied Mathematical Sciences*. Springer Verlag, New York, 1982.

[8] S. Sternberg. Local Contractions and a Theorem of Poincaré. *American Journal of Mathematics*, 79(4):809–824, October 1957.

[9] W. Tucker. A rigorous ODE solver and Smale's 14th problem". *Foundations of Computational Mathematics*, 2:53–117, 2002.