An Economic Analysis of Some Controlled Fertilizer Input—Output Experiments in Michigan
Michigan State University Agricultural Experiment Station
Technical Bulletin

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Issued May 1959
58 pages

Technical Bulletin 269 1959

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AN ECONOMIC ANALYSIS OF SOME CONTROLLED FERTILIZER INPUT-OUTPUT EXPERIMENTS IN MICHIGAN

By W. B. SUNDQUIST and L. S. ROBERTSON, Jr.[1]

INTRODUCTION

MUCH ATTENTION has been devoted recently to the economics of fertilizer use in the U. S. The increased interest in obtaining information regarding fertilizer use is not surprising in view of the magnitude of fertilizer used by farmers throughout the nation. Estimates made in *The 1954 Census of Agriculture* (U. S. Bureau of the Census, 1955) indicate that in 1954 U. S. farmers used $1,024,105,000 worth of fertilizer. Expenditures for farm use in Michigan totalled $31,163,000 in that year.

From 1910 to 1954 fertilizer consumption in the nation showed a substantial secular increase. Total U. S. consumption of elemental nitrogen increased from 46,000 tons in 1910 to 1,868,000 tons in 1954. During the same period, consumption of phosphorus (P_2O_5) increased from 499,000 to 2,228,000 tons. Consumption of potash (K_2O) increased from 211,000 to 1,868,000 tons over the same 44-year period.

Reasons for Increased Fertilizer Use

Three primary reasons account for the rapid increase in the use of commercial fertilizers. First, plant nutrients have become much less expensive in relation to most other farm inputs mainly because of a reduction in bulk and form. Excluding transportation costs, the 1954-55 price of a unit of nitrogen was only about one-third of the adjusted 1920 price (Hignett, 1956). A unit of K_2O was only one-fifth of the adjusted 1920 price in 1954-55 while the adjusted price of a unit of P_2O_5 decreased about 27 percent during this 35-year period.

Second, more information is now available concerning the yield benefits of various crops from applications of primary plant nutrients.

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The authors acknowledge the aid and criticisms of Professors G. L. Johnson, J. F. Davis, and W. A. Cromarty of the Mich. Agr. Expt. Sta.

This information has come in increasing quantities from many sources. Experimental results from agricultural experiment stations and private fertilizer companies have been utilized by farmers.

Agencies and organizations such as the Federal Extension Service, the Tennessee Valley Authority, the National Plant Food Institute, and others have aided in providing farmers with educational materials and demonstrations of the effects of fertilizer on crop yields. In addition, farmers' experiences and those of their neighbors with the use of commercial plant nutrients have caused much of the increase in fertilizer use.

Thirdly, more intensive cropping and higher yields have resulted in a real need for supplying additional plant nutrients to the soil because of the high removal rate.

The Need for Additional Information About Crop Responses to Plant Nutrient Applications

Because of rapid technological change, farming in the U. S. has become a complex operation. Farmers have large fixed and variable resource assets which they must allocate in their farm businesses. As previously indicated, annual fertilizer inputs in excess of a billion dollars represent one of the major variable resource inputs currently used in American agriculture.

If farmers are to allocate their production resources optimally within the farm business, they need information about the prospective earning power of various production resources they may buy. Furthermore, they need this information for various levels of intensity of resource use. No one intensity or level of resource use is likely to be optimal for several farmers at a given time or for a single farmer over a period of time. This is true, first, because of changes in relative prices of resource inputs and production outputs and, second, because of differences in the make-up and amount of resources available to individual farmers.[2]

Because of these complicating factors, the response of crops to applications of the three primary plant nutrients should be measured over a wide range of inputs of the plant nutrients. An assumption essential to this hypothesis is that these nutrients are deficient for production of maximum crop yields over a wide range of applications.

[2]Resource assets vary in total value from farm to farm. In addition, the quantity and quality of fixed resources affect the earning power of variable resources such as fertilizer. Finally, because of forced or voluntary capital or credit rationing restrictions, some farmers find it necessary or desirable to use smaller amounts of variable production resources than would be the case if they had unlimited capital resources.

Typical experimental designs and traditional methods of analyzing plant nutrient input-crop yield output data have been typically lacking[3] in two respects. First, too few levels of plant nutrient applications have been sampled in experiments to allow a reliable statistical estimation of the economically relevant portion of crop response surfaces. Second, the typical type of statistical analysis has been limited to analysis of variance.

This method of analysis is usually used to test whether or not yield differences resulting from different plant nutrient treatments are statistically significant. Using this method of analysis, no basis is available for obtaining a reliable interpolation of yield response between observed treatment levels.

Without estimates of the economically relevant portion of the plant nutrient input-crop yield output surface, it is difficult, and sometimes impossible, to utilize formal deductive economic principles to determine the quantities and combinations of plant nutrients that result in maximum profits.

The experimental designs illustrated and utilized in this publication should provide a more adequate basis for obtaining estimates of fertilizer-crop yield response. In addition, the statistical analysis used allows a comprehensive analysis of the data to determine various economic optima. Subsequent sections of this publication deal with (a) the method of analysis used, (b) specification of a portion of the experimental input-output work conducted by the Michigan Agricultural Experiment Station, (c) an analysis of the data, and (d) an evaluation of the procedures and results.

METHODS OF ANALYSIS

It is the generally recognized purpose of scientific research procedure to establish and verify relationships that are universal to some population.[4] When relating various phenomena in the real world, such as fertilizer-crop yield relationship, two dimensions of such relationships are subject to variance. First, the relationships may vary in the reliability of empirical estimates which can be derived or established for them, i.e., variance in the reliability dimension.

Second, the size of the population to which such derived relation-

[3]"Lacking" is used here to mean inability to facilitate obtaining the type of information which a researcher or a farm manager needs to determine the best plant nutrient applications with varying sets of resources and of fertilizer and crop prices.

[4]Most of these relationships are probability statements about relationships. Thus the universality referred to here does not imply absoluteness of the relationships specified, but rather implies universal applicability to some population of the deductions and inferences made.

ships are universal may vary considerably, i.e., variance in the application dimension. One would not expect, for example, to establish relationships between plant nutrients and crop yields as accurate or as general as those established between the volume and pressure of gas, such as Boyle's law.

However, if we believe that logical, systematic, and describable relationships exist between plant nutrients and crop yields, researchers must postulate the structure of these relationships and obtain empirical estimates of the relationships. In so doing, an optimum level of accuracy in estimates can be defined by equating the cost of obtaining accuracy with its value.

Failure to structure and quantify relationships systematically, when such action is possible, is likely to result in failure to make the best use of scientific procedure in developing a body of information useful to researchers working on this and related problems of soil fertility and/or farm management.

The Concept of Functional Relationships

The principles utilized by economists in determining the best conditions of resource use and production output are stated in numerous publications. However, it is desirable to outline briefly some of the principles of economic theory that can be applied readily to the production relationships in agronomic-economic work. In order to apply effectively the deductive principles of economic theory, the relevant production relationships need to be specified rather systematically or formally.

Agronomists have hypothesized for years that plant nutrients and crop yields are functionally related. These functional relationships, however, have not been readily identified nor easily isolated. The complexity of these functional relationships can be readily ascertained by inspecting the total production function for crop yields of which the plant nutrient-crop yield relationship is only a part or a subfunction.

The yield of a particular crop (Y) in a given time period (t) may be visualized as being some gross product of energy, genetics, and nutrients. This general relationship may be specified as follows:

$$Y_t = f(\text{energy, genetics, nutrients})$$

It can be readily verified from experience that numerous components of the three categories affecting yields are interdependent or interact with each other. For example, the growth characteristics of a plant (classified as genetic) affect the exposure of the plant to sunlight (energy) and its ability to contact and utilize moisture and plant food (nutrients). There are an infinite number of ways in which components of these growth factor groups can be combined to produce varying quantities of product.

In order to be able to obtain useful measures of the effects of varying quantities of the three primary plant nutrients, it is necessary to fix the non-studied growth factors. Factors such as soil type, tillage methods, seed quality, harvesting procedures, fertilizer carriers, time and method of fertilizer applications, etc., are held as constant as feasible in an effort to measure yield response due only to applied plant nutrients.

Some factors, such as rainfall and temperature, cannot be controlled and will vary from one time period to another. Hence it is necessary to study these relationships over a sufficient period in order to obtain a probability distribution of expected responses.

From the foregoing discussion it is apparent that two factors should be kept in mind when evaluating the following work. First, the plant nutrient-crop yield production functions are sub-functions of more general crop growth relationships and therefore do not contain all variables affecting crop yields; and secondly, the derived relationships describe plant nutrient-crop yield relationships only for specific time periods.

In its simplest form, the functional relationship between crop yield and a plant nutrient may be written:

$$Y = f(X)$$

Where Y is the crop yield and X the plant nutrient, in this example, nitrogen. Recognizing that other factors interact with nitrogen, X_1, and are necessary for crop production, we write:

$$Y = f(X_1, X_2, \ldots, X_i, \ldots, X_n)$$

in which X_1 represents nitrogen and X_2 to X_n are factors such as P_2O_5, K_2O, water, temperature, etc. To symbolize that all factors except nitrogen are fixed at some constant level, we write

$$Y = f(X_1/X_2, \ldots, X_i, \ldots, X_n).$$

Furthermore, if all factors that affect crop yields cannot be isolated and specified, we say

$$Y = f(X_1/X_2, \ldots, X_i, \ldots, X_n) + U$$

in which U is an error term representing the unexplained variance of Y (observed yield)[5] from $\hat{Y}$ (predicted yield). If it can be validly assumed that (1) factors that contribute to U, (i.e., unspecified factors) are normally and randomly distributed with respect to the measured variables (in this case X_1) and (2) that the expected value of U is zero, the existence of this unspecified source of yield variance does not bias statistical estimates of the influence of the observed variables on Y.

The specification of the functional relationship between plant nutrients and crop yields, commonly called a production function, has taken different forms over a period of years. Justus Von Liebig's "Law of the Minimum" was an early attempt to specify the form of fertilizer production functions. This formulation postulated that crop yields increased in direct proportion to additions of the nutrient that limited plant growth. Thus, other production factors were assumed to be perfect complements of the limiting factor.

This formulation of the fertilizer-crop yield production function has been rejected because researchers have observed that: (1) production factors are not perfect complements, i.e., a given crop yield may be produced with varying quantities and combinations of applied N, P_2O_5, K_2O, water, etc. provided some minimum quantity of each necessary to produce that crop yield is available and (2) additional inputs of a factor that limits crop yields do not typically result in linear additions to crop yields; rather they result in diminishing additions to crop yields for a time and eventually further additions of the factor cause an actual decrease in total yield.

Choosing Appropriate Production Function Formulations

Since Von Liebig's early formulation, many attempts have been made to use different forms of production functions to describe these input-output relationships. Although numerous types of functions have been formulated, none has been accepted as "best." These various functions have received adequate discussion in other literature (Heady et al., 1955, and Redman and Allen, 1954) and will not be analyzed here.[6]

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[5]If unexplained variance is to be validly attributed solely to components of the error term, U, the specified functional relationship should be the right one, i.e., it should approximate the real world functional relationship.

[6]Historical description of use of production functions in estimating fertilizer-crop yield relations may be found in the following:

Redman, John C., and Stephen Q. Allen (1954). Some interrelationships of economic and agronomic concepts. Jour. of Farm Economics, Vol. 36, 453-465.

Heady, Earl O., John T. Pesek and William Brown (1955). Crop response surfaces and economic optima in fertilizer use. Iowa Agr. Expt. Sta., Res. Bul. 424. Iowa State Col., Ames, Iowa.

There are, however, several criteria that must be satisfied by a particular function if they are to provide a realistic formulation of the input-output relationships between fertilizer inputs and crop outputs. The function should be capable of reflecting successively the following yield responses to added inputs of plant nutrients: (a) yields increasing at a diminishing rate and (b) total yields decreasing.

If the soil is relatively low in initial fertility, an earlier stage of input-output relationships may be present. This is the stage in which yields increase at an increasing rate in response to additional inputs of plant nutrients. In addition, if interaction between plant nutrients is expected, the formulation should include equational variables to specify this interaction.[7]

Final selection of the proper functional form can be facilitated by statistical measures of the goodness of fit of the various functions to the observed data. These tests are essentially of two types: (1) coefficients of multiple correlation and multiple determination or other measures that compare the amount of variance explained by regression with the total amount present in the yield data, and (2) standard errors of the parameters and of the prediction equation.

These measures provide not only a measure of reliability of these statistics but also some insights as to the reliability of derivatives of the function. It was pointed out earlier that these derivatives are necessary in estimating optimum and maximum quantities of plant nutrient inputs. These objective tests may be supplemented by the researcher's examination of the magnitude and distribution of residuals of observed from predicted yield values and his general familiarity with the data.

Some statisticians would contend that statistical estimating procedures are improperly used when the statistics derived are used to compare two or more functions in order to choose the best alternative. They would argue that the proper functional form should be established *a priori* to the fitting by utilizing theory, logic, and experience, and the statistical estimation should be used only to estimate the parameters of the equation of proper form.

If the theory of fertilizer input-output relations was sufficiently developed so that the proper functional form of the production function could be deduced, statistical estimation of the production func-

[7]Such interaction may be incorporated into the functional relationship in several ways. In a sense, it is automatically included in a production function of product form such as an exponential. Special cross-product terms may be included in a polynomial type equation. The point of importance is that it be included so that partial derivatives of yield with respect to individual plant nutrients will reflect the level at which other interacting nutrients are considered.

tion would be greatly simplified. The statistical task would then be that of estimating parameters for the variables in the functional relationship and obtaining reliability measures for these parameters. However, lacking a precise theory as to the proper functional form, various equations must be compared to see which "best" describes the observed relationships.

The conclusive test of whether or not a particular production function formulation is appropriate is its predictive ability over time. This test can be applied only by prediction, further observation, and further prediction.

Production Functions Used In This Analysis

Two basic formulations of fertilizer production functions are utilized and compared in the analysis that follows. These are (1) a polynomial and (2) an exponential type function.

The polynomial equation contains first and second-degree terms for each of the N, P_2O_5, and K_2O variables and first-degree cross-product terms for all nutrients taken two at a time. This formulation is as follows:

$$Y = a + b_1N + b_2N^2 + b_3P + b_4P^2 + b_5K + b_6K^2 + b_7NP + b_8NK + b_9PK.$$

The variables N, P, and K represent per acre applications of N, P_2O_5, and K_2O, respectively. Parameters for the variables are estimated by the technique of least squares. A polynomial equation of this type allows expression of diminishing yields from additional plant nutrient inputs. The estimated parameters are typically expected to be positive for first-degree terms in the equation and negative for second-degree terms. As inputs of plant nutrients become larger the effects of the second-degree terms become more pronounced, and additions to crop yields from succeeding inputs are diminished.

The partial derivatives of yield with respect to the individual plant nutrients are linear provided that this production function is used. The derivative of this function with respect to nitrogen, for example, is:

$$\frac{\partial Y}{\partial N} = b_1 + 2b_2N + b_7P + b_8K$$

Thus the predicted additions to crop yield from additional units of nitrogen is a linear function of nitrogen inputs. The derivative with respect to nitrogen, however, will acquire different values if the level of applications of P_2O_5 and K_2O is changed.

The second formulation of fertilizer production functions utilized in the following analysis is an exponential equation of the Carter-Halter[8] type. This exponential equation is more or less flexible, depending on the magnitude of parameters estimated for the variables. In addition to retaining the curvilinear properties postulated to exist in fertilizer input-crop output relationships, use of this equation facilitates estimation of input-output relationships ranging over all three stages of production, i.e., returns to additional plant nutrients which (1) increase at an increasing rate, (2) increase at a decreasing rate, and (3) become negative.

This formulation in equational form is:

$$Y = aN^{b_1}c_1^{N}P^{b_2}c_2^{P}K^{b_3}c_3^{K}$$

By taking the logarithm of this equation, we acquire an equational form of this function for which the parameters can be estimated by the technique of least squares. The form in which this equation is fitted statistically is:

$$\text{Log } Y = \log a + b_1 \log N + N \log c_1 + b_2 \log P + P \log c_2 + b_3 \log K + K \log c_3.$$

Derivatives of this exponential function are curvilinear. The derivative of the exponential with respect to nitrogen is illustrated below:

$$\frac{\partial Y}{\partial N} = R\,(N^{b_1}c_1^{N}\ln c_1 + c_1^{N}b_1N^{b_1-})$$

where $R = \text{antilog}\,(a + b_2 \log P + \log c_2 + b_3 \log K + K \log c_3)$ the expression of the partial derivative of Y with respect to N may be simplified by factoring out Y which leaves:

$$\frac{\partial Y}{\partial N} = Y\left(\ln c_1 + \frac{b_1}{N}\right)$$

Determining Economic Optima

After obtaining an estimate of the production function for plant nutrients, various optimal combinations of plant nutrients may be determined. If for example, the following equation:

$$Y = a + b_1X_1 + b_2X_1^2 + b_3X_2 + b_4X^2_2 + b_5X_3 + b_6X_3^2$$

[8]The usefulness of this equation as a production function formulation was first noted by H. O. Carter and A. N. Halter. A discussion of the important properties of this function, as well as an explanation of techniques for solving the equation for optimal quantities of the variables are reported by Halter, Carter, and Hocking (1957).

describes the relation of yield to the three plant nutrients X_1, X_2, X_3, then the following procedure is used to find the combination of plant nutrients that will produce the maximum yield.

Taking the partial derivatives of the three nutrients with respect to yield gives:

$$(1)\quad \frac{\partial Y}{\partial X_1} = b_1 + 2b_2X_1$$

$$(2)\quad \frac{\partial Y}{\partial X_2} = b_3 + 2b_4X_2$$

$$(3)\quad \frac{\partial Y}{\partial X_3} = b_5 + 2b_6X_3.$$

Setting each partial derivative equal to zero and solving the three equations simultaneously gives the combination of plant nutrients that will produce the maximum crop yield. To obtain the economically optimal combination of plant nutrients, the prices of plant nutrients, Px_i, where $i = 1, 2, 3$, and the product price, Py, need to be considered. These are considered in the profit equation in which π indicates profit, which follows:

$$\pi = Y\,Py - X_1\,Px_1 - X_2\,Px_2 - X_3\,Px_3 - FC.$$

This equation sets profit equal to the value of the product less the cost of the plant nutrients less fixed costs.

When utilizing unlimited resources, the high-profit combination of plant nutrient occurs when the marginal value product[9] of each nutrient, which is the value of the product produced by an additional unit of the nutrient input, is just equal to the cost of the nutrient input, i.e.,

$$\frac{\partial \pi}{\partial X_i} = O.$$

This occurs where $\frac{\partial Y}{\partial X_i} \cdot Py = Px_i$

Dividing by Py gives $\frac{\partial Y}{\partial X_i} = \frac{Px_i}{Py}$

[9]Marginal value products presented in the analysis that follows do not include a value for residual fertility resulting from applied plant nutrients. But the value of residual fertility should be included in the marginal value product. At present, however, problems of measurement prohibit estimating such values.

which is the equational form of the partial derivatives that can be used readily in solving for high-profit plant-nutrient inputs. Utilizing the partial derivatives of the previous example gives:

$$(1)\quad b_1 + 2b_2X_1 = \frac{Px_1}{Py}$$

$$(2)\quad b_3 + 2b_4X_2 = \frac{Px_2}{Py}$$

$$(3)\quad b_5 + 2b_6X_3 = \frac{Px_3}{Py}$$

Solving these three equations simultaneously gives the optimal combination of plant nutrients for a given set of product and plant nutrient prices. A second order condition is necessary to insure that the combination of nutrients is indeed an optimal one, i.e., one that maximizes profits. The second partial derivatives of yield with respect to the various nutrients, X_i, considered singly and in all possible combinations, must be negative indicating that the marginal value productivity of each of the nutrients was decreasing at the point of optimal combination. Attainment of these second order physical conditions are assured by the law of diminishing returns. In order to assure that these relationships are true in the value dimension as well, we assume the Px_i are constants as is Py.

Characteristics of the Experimental Designs Used

Three factors were given primary consideration in formulating the experimental designs that are illustrated later in this bulletin. These factors were (1) the type of fertilizer input-crop output information needed by farmers, (2) the type of statistical and economic analysis to be utilized, and (3) the resources that were available to conduct the field experimentation and to analyze the data produced.

As previously indicated, farm managers need information as to the quantities and combinations of plant nutrients that will produce maximum profits under different crop and fertilizer prices and price ratios. Thus input-output estimates should be made for all portions of the fertilizer-crop yield production surface that may be of economic relevance.

In order to provide the type of input-output information needed by farmers, it is usually necessary first to obtain statistical estimates of the production surface and then to apply formal economizing principles to determine optimal fertilizer applications. This type of analysis is facilitated by using continuous functions. The use of continuous function analysis assumes, essentially, that by using statistical estimating procedures one can obtain a sufficiently reliable estimate of the economically relevant portions of a fertilizer-crop yield production surface to predict input-output relationships at any relevant point on the surface. This assumption is made not only for the yield of a crop resulting from one plant nutrient variable with others fixed, but also for several plant nutrients in varying combinations.

The use of a continuous function analysis imposes some restrictions on the type of experimental designs to be used. Although no absolute criteria exist for selecting an appropriate experimental design, some criteria can be established relative to the needs of alternative analytical procedures.

Relative to the data needed for analysis of variance, the design for functional analysis should include a more complete specification of the production surface, i.e., observations need to be spread more completely over the entire production surface. Regions of the production surface in which rapidly changing productivity of plant nutrients is to be expected should be sampled adequately to lend sufficient reliability to estimates of the surface and its derivatives in these critical regions.

Particularly critical regions of the production surface are represented by the origin and points of inflection of the fitted function. As reliability measures can be calculated for the estimate of the entire production surface, replications of individual observations are not as valuable for continuous functions as for analysis of variance, i.e., we are not interested so much in measuring significant differences between points on the surface, which requires replicating these points, as we are in obtaining a measure of the reliability of our estimate of the complete production surface and its derivatives.

Omitting any specific surface points from the design does not affect appreciably the reliability of the estimates. The experimental design used for functional analysis can be flexible to the extent of allowing the use of incomplete factorials or other incompletely specified designs.

Very complex functions may be fitted to the data as each added parameter uses only one degree of freedom, which is of little consequence in any experiment containing numerous observations. Complications in calculations, however, impose practical limits on the complexity of functions that can be used. In addition, the more complex the function, the more difficult it is to approximate a reliability measure of the partial derivative of the yield estimate with respect to individual plant nutrients.

Limited funds imposed restrictions on the experimental designs selected. A minimum of six or seven fertilizer treatment levels, including the zero level, appeared to be necessary if an adequate specification of the fertilizer-crop yield production surfaces was to be obtained. With three plant nutrients variable, a complete factorial without replication and with six treatment levels would require 216 individual plots.

With seven treatment levels, the number of required plots would increase to 343 with no replications. Experimental designs including more than 300 plots appeared to be prohibitive both because of the cost of establishing and maintaining the experiments and because of the difficulty in conducting experimental work with proper timeliness.

In addition to the basic input-output data, the experiments produce much additional useful information. The experiments may be used to measure the effects of variable quantities of plant nutrients on both the quality and the chemical composition of crops produced. Data produced in these experiments are useful in relating the quantities of applied plant nutrients with residual fertility build-up in the soil as measured by soil tests. Analysis of these by-product data by analysis of variance techniques may be desirable and appropriate.[10] Primarily for this reason, each experimental design includes a replicated factorial within the overall design.

In conforming to these restrictions, the designs used have the following general characteristics:

(1) Individual observations cover those portions of the production surfaces that are of interest to researchers; (2) as the objective is to estimate the entire surface over the range in which it is of economic importance, the experiments contain a minimum number of replicated plots, thus reducing the need for establishing accurate measurements of individual surface points; (3) the designs involve numerous check plots (plots to which no fertilizer is applied) to establish the origin of fitted functions, i.e., the yield value with no plant nutrients applied; and (4) to the extent possible, intercorrelations among the amounts of nutrients applied have been minimized to facilitate estimation of the

[10]For example, such factors may be characterized by different relationships than the continuous curvilinear relationships expected for plant nutrients and crop yields. In addition, inclusion of a replicated factorial allows preliminary analysis to determine whether or not there are significant differences in studied factors associated with applied plant nutrients.

equational parameters with greater reliability than if the intercorrelations were high. The designs vary somewhat for different experiments, but they may be broadly classified as incomplete factorials.

Application of Method

Several experiments are currently being conducted by the Michigan Agricultural Experiment Station to determine fertilizer input-crop output relationships. The first of these experiments was initiated in the spring of 1954 and additional experiments have been conducted since. Currently, input-output experiments are underway for potatoes, corn, wheat, oats, alfalfa, and field beans.

In addition, an input-output study for sugar beets was initiated in 1957. Experimental work is conducted by the Department of Soil Science and with the Department of Agricultural Economics and the Farm Economics Research Division, Agricultural Research Service, U. S. Department of Agriculture cooperating on the design of the experiments and analysis of the data.

The Oats, Wheat, Alfalfa, and Corn Rotation[11]

In the spring of 1955, an experiment was initiated for a rotation of oats, wheat, alfalfa, and corn. This experiment is located at two sites in Kalamazoo and Calhoun Counties on a Kalamazoo sandy loam soil. This is a light upland soil, which tends to be somewhat droughty and relatively low in natural fertility. Each crop of the rotation is grown each year. Thus there are four fields each with the same experimental design.

The experiment includes the three primary plant nutrients—nitrogen, phosphoric acid, and potash—in varying combinations. Six treatment levels, including a zero application level, are included in the experiment for each plant nutrient. Measured in pounds per acre, these treatment levels are:

N —	0	20	40	80	160	240
P_2O_5 —	0	40	80	160	320	480
K_2O —	0	20	40	80	160	240

Ninety-one individual surface points are sampled, 27 of which are replicated twice in a 3 x 3 x 3 factorial at the second, fourth, and sixth treatment levels. There are 11 replications of the check (0-0-0) treatment.

[11]This experiment has been conducted with the aid of funds and other resources from the National Plant Food Institute and the Davison Chemical Corporation.

There are 130 plots in each of the four fields in the experiment.

Individual plots are 50 by 14 feet in size, making a total area per plot of about 1/62.5 of an acre. The 14-foot width facilitates use of a 7-foot grain drill for application of fertilizer and seed, and a 7-foot self-propelled combine for harvesting operations. Most fertilizer applications are made by broadcasting the fertilizer, either mechanically or by hand, prior to plowing the ground and preparatory to planting the crop. Two notable exceptions are: (1) The first level of applied P_2O_5 (40 pounds per acre) is applied in the row at planting time as a starter fertilizer and (2) the alfalfa crop is fertilized by top-dressing in the spring. The design for this experiment is shown in detail in Table 1.

Continuous Corn

An experiment in which corn is grown in continuous culture was initiated in Tuscola county in 1956. This experiment is located on a Wisner clay loam soil, which is one of the heavier, more productive soils occurring in the state. The experiment contains 204 individual plots representing 139 surface points. Included in the design is a 3 x 3 x 3 factorial replicated three times, including observations at the second, fourth, and sixth treatment levels.

In addition, there are eight check plots. Inclusion of the triplicated factorial allows a limited study of yields and other experimental data by analysis of variance techniques. In pounds per acre, the seven treatment levels for the three plant nutrients in this experiment are as follows:

N	— 0	20	40	80	160	240	320
P_2O_5	— 0	40	80	160	320	480	640
K_2O	— 0	20	40	80	160	240	320

Individual plots are 55 by 14 feet in size, allowing 4 rows of corn spaced 42 inches apart to be grown on each plot. The design for this experiment is shown in detail in Table 2.

Field Beans, Wheat, and Corn Rotation[12]

An intensive rotation of field beans, wheat, and corn was initiated in Gratiot county in 1955. Corn was produced on these plots in 1955 and field beans in 1956. The experiment is located on a Sims loam

[12]This experiment is cooperative between the Mich. Agr. Expt. Sta. and the Tennessee Valley Authority (T.V.A.). Robert D. Munson and other personnel from the Agricultural Economics Branch, Division of Agricultural Relations of the T.V.A. have been actively engaged in planning and carrying out this experiment as well as other closely related experimental work.

TABLE 1—Experimental design for the oats, wheat, alfalfa, and corn rotation

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
0	0	0	11	80	160	80	2
0	40	20	1	80	160	240	2
0	160	0	1	80	320	40	1
0	160	80	1	80	320	160	1
0	160	240	1	80	320	240	1
0	480	80	1	80	480	0	1
0	480	240	1	80	480	20	2
20	0	20	1	80	480	80	2
20	40	0	1	80	480	160	1
20	40	20	2	80	480	240	2
20	40	80	2	160	40	40	1
20	40	240	2	160	40	160	1
20	80	40	1	160	80	20	1
20	80	160	1	160	80	40	1
20	160	20	2	160	80	80	1
20	160	80	2	160	80	240	1
20	160	240	2	160	160	40	1
20	320	40	1	160	160	160	1
20	320	160	1	160	160	240	1
20	480	20	2	160	320	20	1
20	480	80	2	160	320	80	1
20	480	240	2	160	320	160	2
40	40	40	1	160	320	240	1
40	40	160	1	160	320	40	1
40	80	20	1	160	480	80	1
40	80	40	2	160	480	160	1
40	80	80	1	160	480	240	1
40	80	240	1	240	0	80	1
40	160	40	1	240	0	240	1
40	160	160	1	240	40	20	2
40	320	20	1	240	40	80	2
40	320	80	1	240	40	240	2
40	320	240	1	240	80	160	1
40	480	40	1	240	160	20	2
40	480	160	1	240	160	80	2
80	0	0	1	240	160	240	2
80	0	80	1	240	320	0	1
80	0	240	1	240	320	40	1
80	40	20	2	240	320	160	1
80	40	80	2	240	320	240	1
80	40	240	2	240	480	0	1
80	80	40	1	240	480	20	2
80	80	160	1	240	480	80	2
80	160	0	1	240	480	160	1
80	160	20	2	240	480	240	2

TABLE 2—Experimental design for the continuous corn experiment

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
0	0	0	8	40	320	320	1
0	0	40	1	40	480	40	1
0	40	240	1	40	480	80	1
0	80	0	1	40	480	240	1
0	80	40	1	40	640	20	1
0	160	320	1	40	640	160	1
0	320	160	1	40	640	320	1
0	480	20	1	80	0	0	1
0	640	80	1	80	0	160	1
0	640	320	1	80	40	20	3
20	40	20	3	80	40	80	3
20	40	80	3	80	40	240	3
20	40	160	1	80	80	40	1
20	40	240	3	80	80	160	1
20	80	20	1	80	80	240	1
20	80	80	1	80	160	20	3
20	80	240	1	80	160	80	3
20	160	20	3	80	160	240	3
20	160	40	1	80	160	320	1
20	160	80	3	80	320	0	1
20	160	240	3	80	320	40	1
20	320	20	1	80	320	160	1
20	320	160	1	80	480	20	3
20	320	320	1	80	480	40	1
20	480	20	3	80	480	80	3
20	480	40	1	80	480	240	3
20	480	80	3	80	640	80	1
20	480	240	3	80	640	320	1
20	640	160	1	160	0	20	1
20	640	320	1	160	0	80	1
40	0	0	1	160	40	0	1
40	0	40	1	160	40	80	1
40	40	20	1	160	40	240	1
40	40	40	1	160	80	40	1
40	40	80	1	160	80	160	1
40	40	160	1	160	160	0	1
40	40	320	1	160	160	20	1
40	80	0	1	160	160	80	1
40	80	40	2	160	160	160	1
40	80	240	1	160	160	240	1
40	160	20	1	160	160	320	1
40	160	80	1	160	320	40	1
40	160	160	1	160	320	80	1
40	160	240	1	160	320	160	2
40	320	20	1	160	480	20	1
40	320	40	1	160	480	80	1
40	320	80	1	160	480	320	1
40	320	160	1	160	640	80	1

TABLE 2—Concluded

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
160	640	240	1	320	0	80	1
240	0	0	1	320	40	20	1
240	0	40	1	320	40	240	1
240	40	20	3	320	80	40	1
240	40	80	3	320	80	80	1
240	40	240	3	320	80	160	1
240	80	0	1	320	80	240	1
240	80	40	1	320	80	320	1
240	80	160	1	320	160	0	1
240	80	320	1	320	160	40	1
240	160	20	3	320	160	320	1
240	160	80	3	320	320	40	1
240	160	240	3	320	320	80	1
240	320	40	1	320	320	160	1
240	320	160	1	320	320	240	1
240	320	320	1	320	480	20	1
240	480	20	3	320	480	160	1
240	480	80	3	320	480	320	1
240	480	240	3	320	640	40	1
240	640	40	1	320	640	80	1
240	640	160	1	320	640	240	1
240	640	320	1	320	640	320	2

soil, a heavy productive soil that can be cropped intensively without risk of erosion. The seven treatment levels for the three plant nutrients are identical to those in the continuous corn experiment. The treatments in pounds per acre of applied plant nutrients are:

N	— 0	20	40	80	160	240	320
P_2O_5	— 0	40	80	160	320	480	640
K_2O	— 0	20	40	80	160	240	320

The experiment is an incomplete factorial. It consists of 193 individual surface points, of which 27 are replicated twice in a 3 x 3 x 3 factorial at the first, fourth, and sixth treatment levels. There are 11 check plots in the basic experimental design, which contains a total of 233 individual plots. Extra plots were included in the experiment for purposes of other analyses bringing the total number of plots to 258. Individual plots in this experiment are 50 by 14 feet in size.

The design for this experiment includes a more complete specification of the production surface than any other experiment and the total of 193 different surface points exceeds that of any other experiment currently underway. The experimental design for this experiment is shown in Table 3.

TABLE 3—Experimental design for the beans, wheat, and corn rotation

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
0	0	0	11	20	480	20	1
0	0	20	1	20	480	40	1
0	0	40	1	20	480	160	1
0	0	80	1	20	480	320	1
0	0	160	1	20	640	20	2
0	0	240	1	20	640	40	1
0	0	320	1	20	640	80	1
0	40	0	1	20	640	160	2
0	40	240	1	20	640	240	1
0	80	0	1	20	640	320	2
0	80	40	1	40	0	0	1
0	160	0	1	40	0	40	1
0	160	320	1	40	40	20	1
0	320	0	1	40	40	80	1
0	320	160	1	40	40	160	1
0	480	0	1	40	40	320	1
0	480	20	1	40	80	0	1
0	640	0	1	40	80	40	2
0	640	80	1	40	80	240	1
0	640	320	1	40	80	320	1
20	0	0	1	40	160	20	1
20	40	20	2	40	160	80	1
20	40	40	1	40	160	160	1
20	40	80	1	40	160	320	1
20	40	160	2	40	320	20	1
20	40	240	1	40	320	80	1
20	40	320	2	40	320	160	1
20	80	20	1	40	320	320	1
20	80	80	1	40	480	40	1
20	80	160	1	40	480	80	1
20	80	240	1	40	480	240	1
20	80	320	1	40	640	20	1
20	160	20	1	40	640	80	1
20	160	40	1	40	640	160	1
20	160	160	1	40	640	320	1
20	160	240	1	80	0	0	1
20	160	320	1	80	0	320	1
20	320	20	1	80	40	20	1
20	320	40	1	80	40	80	1
20	320	80	1	80	40	160	1
20	320	160	2	80	40	320	1
20	320	320	2	80	80	40	1

TABLE 3—Continued

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
80	80	240	1	160	640	80	1
80	80	320	1	160	640	160	2
80	160	20	1	160	640	240	1
80	160	80	2	160	640	320	2
80	160	160	1	240	0	0	1
80	320	20	1	240	40	20	1
80	320	80	1	240	40	80	1
80	320	160	1	240	40	160	1
80	320	320	1	240	40	320	1
80	480	40	1	240	80	40	1
80	480	240	1	240	80	320	1
80	480	320	1	240	160	20	1
80	640	0	1	240	160	80	1
80	640	20	1	240	160	160	1
80	640	40	1	240	160	240	1
80	640	160	1	240	320	20	1
80	640	240	1	240	320	80	1
80	640	320	1	240	480	160	1
160	0	0	1	240	320	640	1
160	0	160	1	240	480	40	1
160	40	20	2	240	480	80	1
160	40	40	1	240	480	240	2
160	40	80	1	240	640	20	1
160	40	160	2	240	640	80	1
160	40	240	1	240	640	160	1
160	40	320	2	240	640	320	1
160	80	20	1	320	0	0	1
160	80	80	1	320	0	80	1
160	80	160	1	320	0	320	1
160	80	240	1	320	40	20	2
160	80	320	1	320	40	40	1
160	160	20	1	320	40	80	1
160	160	40	1	320	40	160	2
160	160	160	1	320	40	320	1
160	160	320	1	320	40	320	2
160	320	0	1	320	80	20	1
160	320	20	2	320	80	80	1
160	320	40	1	320	80	160	1
160	320	80	1	320	80	240	1
160	320	160	2	320	160	0	1
160	320	240	1	320	160	20	1
160	320	320	2	320	160	40	1
160	480	20	1	320	160	160	1
160	480	40	1	320	160	320	1
160	480	160	1	320	320	20	2
160	480	320	1	320	320	40	11
160	640	20	2	320	320	80	1
160	640	40	1	320	320	160	2

TABLE 3—Concluded

Plant nutrients (pounds per acre)			No. of plots	Plant nutrients (pounds per acre)			No. of plots
N	P_2O_5	K_2O		N	P_2O_5	K_2O	
320	320	240	1	320	640	20	2
320	320	320	2	320	640	40	1
320	480	20	1	320	640	80	1
320	480	40	1	320	640	160	2
320	480	160	1	320	640	240	1
320	480	320	1	320	640	320	2
320	640	0	1				

The Total Fertilizer Input-Output Experimental Program

Only a part of the input-output studies conducted by the Michigan Agricultural Experiment Station is reported here. Additional experiments are underway for potatoes and sugar beets.[13] The three experiments outlined above contain almost 1,000 individual plots. In addition to the basic input-output determinations, soil-test measures are acquired for each plot and a rather detailed project is being conducted in which chemical determinations are made of the composition of plant tissue as well as the chemical composition and quality of crops produced. Relative to experimental work undertaken elsewhere, this is an extensive and detailed project.

ANALYSIS OF THE DATA

The oat, wheat, alfalfa, and corn rotation experiment was initiated in 1955 and data collected for the first two years are reported here. Only two harvested crops were produced in 1955, as alfalfa and wheat stands could not be established in time for harvest. Field data were acquired for both corn and oats in 1955 and all four crops were produced in 1956. Because of a heterogeneous stand of alfalfa, no data were acquired for that crop in 1956.

Analysis of the Data on Oats

Oats were produced on two of the experimental sites in Calhoun and Kalamazoo counties in 1955.[14] Preliminary graphic analysis of

[13]Many other experiments conducted by the Mich. Agr. Expt. Sta. also produce valuable information on the responses of various crops to applied plant nutrients.
[14]Eaton was the variety of oats produced on both sites in 1955.

these data indicated that the variance present in the yield data was not associated with variance in the quantities of applied plant nutrients. This hypothesis was further substantiated by fitting a polynomial equation to the data.

None of the variables in this equation had estimated parameters that differed significantly from zero. Apparently, weather was the main determinant that limited crop yields during the 1955 crop growing season. Unfavorable weather, largely the result of a late summer drought, prevented increases in crop yield that might have occurred with increased applications of plant nutrients.

Yield data for oats were acquired again in 1956.[15] Preliminary graphic analysis of these data indicated that positive relationships existed between yields of oats and applied N and P_2O_5. Furthermore, these relationships appeared to be curvilinear, reflecting diminishing returns to inputs of plant nutrients.

The first formulation of the functional relationship that was attempted for the 1956 data was a nine-term polynomial. This formulation containing the estimated parameters is shown in equation I. Values listed below the estimated parameters and included in parentheses are standard errors of the respective parameters. N, P, and K again represent per acre applications of N, P_2O_5, and K_2O, respectively, as is the case in all equations unless otherwise indicated.

$$\text{Equation (I): } \hat{Y}_o = 43.326378 + \underset{(.05115313)}{.40112190\,N} - \underset{(.0019075)}{.00130761\,N^2}$$

$$- \underset{(.02579697)}{.00650205\,P} + \underset{(.00004775)}{.00000534\,P^2} + \underset{(.05196548)}{.06186818\,K} - \underset{(.00019148)}{.00010387\,K^2}$$

$$+ \underset{(.00006495)}{.00000068\,NP} - \underset{(.00013020)}{.00010905\,NK} + \underset{(.00006430)}{.00007542\,PK}$$

The adjusted coefficient of multiple correlation for this equation was 0.690. The coefficient of multiple determination indicated that about 48 percent of the variance in yields of oats was associated with regression. Estimated coefficients for the nitrogen variables were significant at the 1 percent probability level. None of the coefficients for other variables were significant at the 10 percent level of probability.[16]

[15]The variety of oats produced in 1956 was Craig.

[16]Testing the significance of coefficients for individual variables in an equation that contains more than one variable for a given plant nutrient is a practice of limited usefulness. The related variables in such an equation as N, N^2, log N, etc., are obviously highly correlated. Estimates of individual parameters may be subject to large standard errors reflecting these high intercorrelations. One might conclude that as individual parameters are not statistically significant, no significant effects are present. This conclusion might well be fallacious. If the aggregate effect of all variables representing a particular plant nutrient could be tested for significance, the test might indicate a significant aggregate effect. This situation illustrates an inadequacy in current statistical testing procedures. When (1) two or more independent variables in a production function occur in product form or (2) more than one

The second formulation of the 1956 oat data was an exponential function of the Carter-Halter type. This equation with estimated parameters is shown in equation II.

Equation (II): $\text{Log } \hat{Y}_o = 1.57315152 + \underset{(.02022815)}{.16475028} \log N + \underset{(.00015046)}{.00057687} N$
$- \underset{(.01648010)}{.02441092} \log P + \underset{(.00006694)}{.00009610} P - \underset{(.02017332)}{.00634345} \log K + \underset{(.00014714)}{.00021757} K$

The coefficient of multiple correlation for this equation was 0.760. The coefficient of multiple determination indicated that about 58 percent of the variance in oat yields was associated with regression.

In this equation, coefficients for nitrogen variables were significant at the 1 percent probability level, whereas the coefficients for other variables were not statistically significant.

Interpretation of the Statistical Results

It seems desirable to discuss several aspects of the two alternative production function formulations presented here. A comparison of the production surfaces generated by the two functions is of particular interest. In addition, we may compare the combinations of plant nutrients that (1) maximize yields and (2) maximize profits under various plant nutrient and crop prices.

A comparison of oat yields predicted from the two functions for selected combinations of applied plant nutrients is shown in Table 4. Observations from 28 combinations of plant nutrients are included. These include averages of observations from all 27 pairs of plots in the 3 x 3 x 3 replicated factorial, in addition to the average yield from all checked plots.

Statistical measures derived for the equations, including the coefficient of multiple correlation and standard errors of the regression coefficients,[17] indicate that the exponential is a slightly, but not significantly, more appropriate formulation than the polynomial. Inspection

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variable is used to measure the effects of a particular plant nutrient, it would be desirable to obtain a reliability measure on the derivative of crop yield with respect to individual plant nutrients. Such derivatives are necessarily utilized in determining marginal nutrient effects and consequently optimal applications of plant nutrients. A satisfactory procedure for computing reliability measures for such derivatives has not yet been developed, but it is a critical need in much analytical production economics work. Because of this limitation of the standard errors of the regression coefficients, some variables with nonsignificant parameters are left in the functions provided the sign of the parameters are consistent with expectations.

[17]Measures such as correlation coefficients and standard errors of regression coefficients and equations are not without some limitations in comparing these two functions. The observations, and hence the variances, of the variables are not readily comparable as in one instance they are in real numbers and in the other in logarithms. The real numbers and logarithms, although they bear a consistent monotonic relationship to each other, do not maintain a relationship of equivalence or of constant ratios. Hence, the listed statistical measures should not be given an absolute interpretation for comparative purposes, i.e., they should, instead, serve as a basis for a rough comparison.

of the residual values, $(Y_i - \hat{Y}_i)$, for both functions provides little basis for choice between functions, as the individual residual values of the two functions are about equally dispersed with respect to magnitude and direction.[16a]

Some additional insight into the appropriateness of the two alternative functions may be gained by comparing the derivatives of these functions with respect to their correspondence to input-output relationships postulated to exist in accordance with currently held theory.

[16a] These residuals are shown in columns 7 and 8 in Table 4.

TABLE 4—Observed and estimated oat yields, 1956

Treatment (pounds per acre)			Predicted yield (bu. per acre)		Observed yield(*a*) (bu. per acre)	Residual(*b*) $(Yi-\hat{Y}i)$	
N	P_2O_5	K_2O	Exp.(*c*)	Poly.		Exp.	Poly.
0	0	0	37.4	43.3	38.7	1.3	−4.6
20	40	20	56.7	51.8	67.5	10.8	15.7
20	40	80	58.9	54.9	55.1	−3.8	0.2
20	40	240	64.3	59.6	63.5	−0.8	3.9
20	160	20	56.3	51.3	51.0	−5.3	−0.3
20	160	80	58.5	54.9	70.4	11.9	15.5
20	160	240	63.8	61.2	57.9	−5.9	−3.3
20	480	20	58.8	50.8	56.4	−2.4	5.6
20	480	80	61.1	55.9	60.0	−1.1	5.1
20	480	240	66.7	66.0	60.6	−6.1	−5.4
80	40	20	65.7	67.9	75.8	10.1	7.9
80	40	80	68.4	70.6	72.1	3.7	1.5
80	40	240	74.6	74.3	84.2	9.6	9.9
80	160	20	65.3	67.4	76.9	11.6	9.5
80	160	80	67.9	70.7	49.4	−18.5	−21.3
80	160	240	74.0	75.8	71.0	−3.0	−4.8
80	480	20	68.2	66.9	61.2	−7.0	−5.7
80	480	80	70.9	71.7	72.3	1.4	0.6
80	480	240	77.4	80.6	84.3	6.9	3.7
240	40	20	63.7	64.8	71.7	8.0	6.9
240	40	80	66.5	66.2	66.6	0.1	0.4
240	40	240	72.3	67.3	61.7	−10.6	−5.6
240	160	20	63.2	64.3	57.2	−6.0	−7.1
240	160	80	65.8	66.6	66.2	0.4	−0.4
240	160	240	71.7	68.9	69.2	−2.5	0.3
240	480	20	66.1	63.9	76.2	10.1	12.3
240	480	80	68.7	67.6	72.3	3.6	4.7
240	480	240	75.0	73.7	80.6	5.6	6.9

(*a*) The observed yield for the 0-0-0 treatment is an average of yields from 11 plots; all other observed yields are averages of two plots.

(*b*) Residuals are deviations of predicted yields from average observed yields.

(*c*) In computing $\hat{Y}i$ for zero treatments of plant nutrients using the exponential equation, inputs of a single pound of N, P_2O_5, and K_2O were used. This introduces a slight upward bias in the predicted yield but overcomes the problem of having $\hat{Y}i = O$ when any of the treatments is zero. This procedure is utilized in all of the following analyses when computing $\hat{Y}i$ from exponential equations.

In addition, the derivatives are used to calculate plant nutrient combinations which produce maximum yields and maximum profits. Maximum yields occur when the first order partial derivatives of the functions are equal to zero. Maximum profits occur when the partial derivatives with respect to individual plant nutrients are equal to the plant nutrient-crop price ratios.

As most of the variance explained by regression is associated with the nitrogen variable, the derivatives of the functions with respect to nitrogen, $\frac{\partial Y_o}{\partial N}$, are of particular interest. These derivatives are represented by Equations III and IV. All derivatives are taken for a unit (one pound) change in plant nutrients. The derivative of the polynomial is given by the following expression:

$$\frac{\partial Y_o}{\partial N} = b_1 + 2b_2 N + b_7 P + b_8 K$$

Substituting in the estimated parameters from Equation (I) gives

Equation III: $\frac{\partial Y_o}{\partial N} = .40112190 - 2(.00130761)N + .00000068 P$ $- .00010905 K.$

The expression of the partial derivative of the exponential is given by the following expression:

$$\frac{\partial Y_o}{\partial N} = Y_o (\ln c_1 + \frac{b_1}{N}).$$

Substituting in the estimated parameters from Equation II gives Equation IV:

$$\frac{\partial \hat{Y}_o}{\partial N} = \hat{Y}_o (-.00132853 + \frac{.16475028}{N})$$

The partial derivatives of the two functions with respect to N are shown in Table 5, with P_2O_5 and K_2O fixed at three different levels—20-40, 80-160 and 240-480 pounds per acre, respectively. These derivatives are also shown in Figure 1.

Derivatives of the exponential function are larger at small nitrogen inputs than they are for the polynomial function. It is probable that the exponential generates a production surface that rises too rapidly with small nitrogen inputs. If this is true, the derivatives are probably too responsive to small changes in inputs. This phenomenon is due partly to the fact that when $X_i = O$, $Y = O$. The function may still be quite reliable over the range of moderate inputs.

The derivative of the exponential function is 1.46 bushels per pound of nitrogen with a nitrogen input of 5 pounds and decreases to 0.78 bushel when 10 pounds are applied. These values of the derivative appear to be excessively high. However, the derivatives of the exponential type function are not restricted to a linear function of plant nutrient inputs as is the case with a polynomial that contains only first- and second-degree terms.

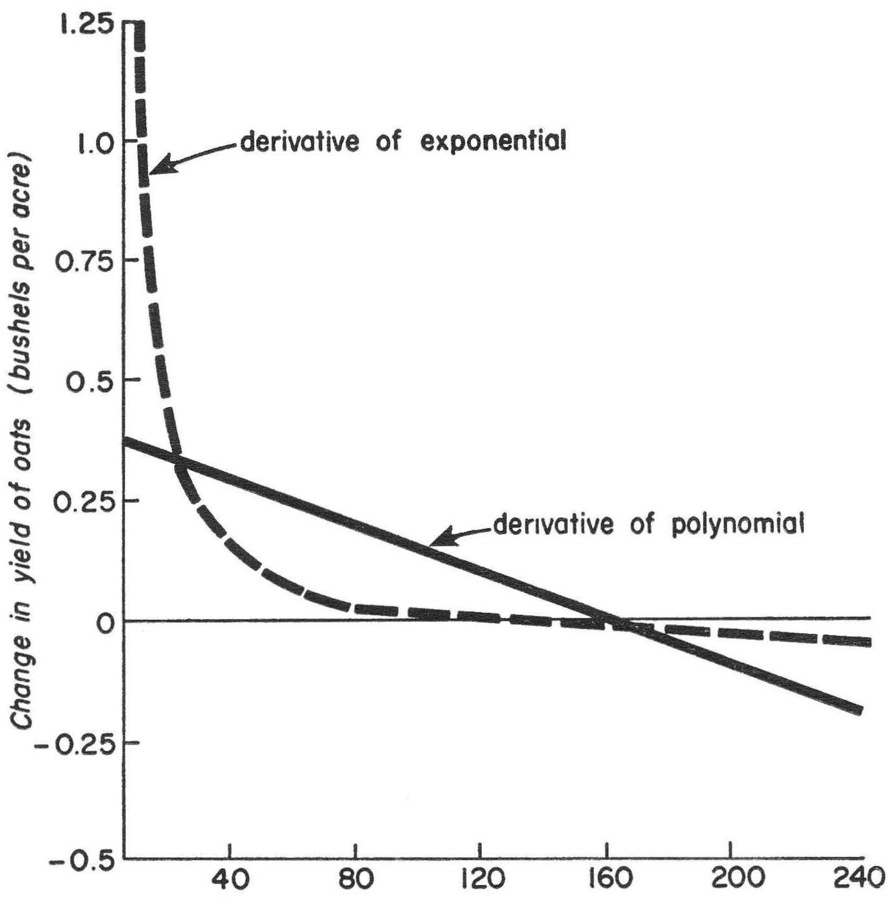

Fig. 1. Partial derivatives of polynominal and exponential functions for oats with respect to nitrogen.

The linearity restriction on the derivatives of a polynomial can be overcome by modifying the formulation to include variables raised to fractional powers, e.g., powers such as 3/2, 1/2, etc. and/or by adding variables that involve powers greater than 2. The statistical fit might not be improved by such a modification, but derivatives would be allowed to become a curvilinear function of additional plant nutrients. Further experimentation with the use of fractional powered and more complex polynomials, as well as additional inspection of the derivatives of these functions is needed and is being conducted.

High Profit Combinations of Plant Nutrients

The optimal amount of plant nutrients to apply, as stated previously, is a function not only of the productivity of applied plant nutrients but also of the prices of plant nutrients and crops. To obtain the combination of applied plant nutrients that will maximize yields, partial derivatives of yield with respect to all plant nutrients are set equal to zero and solved simultaneously.

For the polynomial equation, the maximum estimated yields were obtained with 153 pounds of N, a slightly negative quantity of P_2O_5, and 0.1 pound of K_2O. The estimated amounts of P_2O_5 and K_2O resulting in maximum yields are neither statistically nor economically significantly different from zero. Maximum estimated yields using the exponential equation occur with nitrogen inputs of about 130 pounds per acre.

To solve for optimal fertilizer applications with different crop and fertilizer prices, the derivatives are set equal to the plant nutrient-crop price ratios. As nitrogen was the only applied plant nutrient that had significant effect on oat yields, it is the only plant nutrient contained in the optimal applications. Estimated optimum application rates using the two alternative production function formulations are shown in Table 5-6.

The disparity between optimum treatment rates is unusually large. This is an exceptional case, however, since in no other experiment were the differences so large. Optimum plant nutrient applications are more responsive to price changes when the polynomial is used than when the exponential equation is used. Using either production function, however, only moderate applications of nitrogen are profitable, assuming typical fertilizer-oats price ratios.

Although no applications of P_2O_5 and K_2O are indicated to be profitable by the production function predictions of oat yields alone,

TABLE 5—Changes in oats yields resulting from unit changes in nitrogen applications

Treatment level of P_2O_5 and K_2O(*a*)	Nitrogen treatment level (pounds per acre)	Derivative of polynomial (bu. per acre)	Derivative of exponential(*b*) (bu. per acre)
1	20	.347	.393
1	40	.294	.174
1	80	.190	.050
1	120	.085	.005
1	160	−.019	−.018
1	200	−.124	−.031
1	240	−.229	−.039
2	20	.341	.406
2	40	.288	.180
2	80	.184	.051
2	120	.079	.005
2	160	−.013	−.018
2	200	−.130	−.032
2	240	−.235	−.040
3	20	.323	.462
3	40	.270	.205
3	80	.166	.059
3	120	.061	.006
3	160	−.043	−.021
3	200	−.148	−.037
3	240	−.253	−.046

(*a*) Nitrogen is varied with P_2O_5 and K_2O fixed at three levels: (1) 40-20, (2) 160-80, and (3) 480-240 pounds per acre, respectively.
(*b*) The derivatives are those resulting from an additional pound of nitrogen.

TABLE 6—Predicted optimal nitrogen application(a) with varying oats and nitrogen prices

Price of oats (per bushel)	Price of nitrogen (per pound)							
	$0.09		$0.11		$0.13		$0.15	
	Poly.	Exp.	Poly.	Exp.	Poly.	Exp.	Poly.	Exp.
$0.60	96	46	83	40	70	36	58	32
$0.70	104	50	93	44	82	39	71	36
$0.80	110	55	101	48	91	43	82	39
$0.90	115	59	107	52	98	47	90	43
$1.00	119	63	111	56	104	51	96	46

(*a*) Optimal nitrogen applications do not vary appreciably with changes in the level of P_2O_5 and K_2O applications over the range of observed applications. These application rates were computed using 1956 experimental data only and are not to be interpreted as being recommended treatment rates.

other considerations, such as establishment of grass or legume seedings in the oat crop, probably would have required their application in small or moderate amounts.

Analysis of the Wheat Data

Wheat[18] was produced on the Kalamazoo county experimental site in 1956. The yield data produced in this experiment were analyzed in the same way as they were for oats. The original function fitted to the wheat data was a nine variable polynomial. This formulation with estimated parameters is shown in Equation V.

Equation (V): $\hat{Y}_w = 28.53873032 + \underset{(.01695990)}{.08598469N} - \underset{(.00006324)}{.00022084N^2} +$

$\underset{(.00855302)}{.01637506\,P} - \underset{(.00001583)}{.00003511\,P^2} + \underset{(.01722924)}{.00857080\,K} + \underset{(.00006348)}{.00002132\,K^2} +$

$\underset{(.00002153)}{.00001902\,NP} - \underset{(.00004456)}{.00007994\,NK} + \underset{(.70926011)}{.00001512\,PK}$

The adjusted coefficient of multiple correlation for this equation was 0.66 and the coefficient of multiple determination indicated that about 44 percent of the variance in yield was associated with variance in applied plant nutrients. As with oats, only the estimated parameters for the nitrogen variables were statistically significant at the 1 percent probability level. However, the phosphoric acid variables, P and P^2, were significant at the 5 percent probability level.

A Carter-Halter type exponential function was also fitted to the wheat data. The results of this fit are shown in Equation VI.

Equation (VI): $\text{Log}\,\hat{Y}_w = 1.44504717 + \underset{(.01174513)}{.02263580\log N} +$

$\underset{(.00008736)}{.00017268\,N} + \underset{(.00956889)}{.01657636\log P} - \underset{(.00003886)}{.00001737\,P} +$

$\underset{(.01171329)}{.00127968\log K} + \underset{(.00008543)}{.00010823\,K}$

The adjusted coefficient of multiple correlation for this equation was 0.65. The adjusted coefficient of multiple determination indicated that about 43 percent of the variance in crop yields was associated with variance in the amounts of applied plant nutrients. The first three estimated coefficients in this equation were significant at the

[18]Cornell 595 was the variety of wheat produced in 1956.

10 percent probability level but not at the 5 percent level. The last three coefficients in the equation were not statistically significant.

On the basis of the reliability measures for the regression coefficients, the polynomial equation appears to be the better production function formulation. A comparison of observed yields with yields estimated by using the two functions is shown in Table 7. As was the case for the data on oats, the tabular comparison includes observations and predictions for 28 combinations of applied N, P_2O_5, and K_2O. The observed yield values are averages of two replications for all treatments except the check (0-0-0) treatment, which is an average of 11 replications.

TABLE 7—Observed and estimated wheat yields, 1956

Treatment (pounds per acre)			Predicted yield (bu. per acre)		Observed yield(*a*) (bu. per acre)	Residual(*b*) ($Yi—\hat{Y}i$)	
N	P_2O_5	K_2O	Exp.	Poly.		Exp.	Poly.
0	0	0	27.9	28.5	28.2	0.3	−0.3
20	40	20	32.2	30.9	29.5	−2.7	−1.4
20	40	80	32.7	31.5	29.3	−3.4	−2.2
20	40	240	34.1	33.8	34.6	0.5	0.8
20	160	20	32.8	32.1	31.2	−1.6	−0.9
20	160	80	33.3	32.8	30.4	−2.9	−2.4
20	160	240	32.7	35.4	35.1	3.4	−0.3
20	480	20	33.0	30.4	32.1	−0.9	1.7
20	480	80	33.5	31.4	31.9	−1.6	0.5
20	480	240	34.9	34.8	34.9	0.0	0.1
80	40	20	34.0	34.7	37.5	3.5	2.8
80	40	80	34.6	35.0	34.9	0.3	−0.1
80	40	240	36.1	36.6	37.2	1.1	0.6
80	160	20	34.6	36.1	40.9	6.3	4.8
80	160	80	35.2	36.5	36.7	1.5	0.2
80	160	240	36.7	38.3	36.5	−0.2	−1.8
80	480	20	34.8	34.7	28.7	−6.1	−6.0
80	480	80	35.4	35.4	39.0	3.6	3.6
80	480	240	36.9	38.0	38.6	1.7	0.6
240	40	20	37.2	37.0	36.6	−0.6	−0.4
240	40	80	37.8	36.6	35.1	−2.7	−1.5
240	40	240	39.4	36.1	35.1	−4.3	−1.0
240	160	20	37.8	38.8	42.1	4.3	3.3
240	160	80	38.5	38.4	39.3	0.8	0.9
240	160	240	40.1	38.2	38.4	−1.7	0.2
240	480	20	38.1	38.4	42.9	4.8	4.5
240	480	80	38.7	38.3	38.2	−0.5	−0.1
240	480	240	40.3	38.8	38.8	−1.5	0.0

(*a*) The observed yield is the average of two replications except for the check (0-0-0) treatment, which is the average of 11 replications.

(*b*) Residuals are the difference between average observed yields and estimated yields.

The coefficients of multiple correlation and determination indicated that the two functions were about equally effective in explaining variance in wheat yields. Inspection of the residuals for the two functions ($Y_i - \hat{Y}_i$) further substantiates the conclusion that the two functions produce about equally good fits. These residuals are shown in columns 7 and 8 of Table 7.

Derivatives of the two functions with respect to N and P_2O_5 are presented in Tables 8 and 9 and in Figs. 2 and 3. The derivatives

TABLE 8—Changes in wheat yields resulting from unit changes in nitrogen applications

Treatment level of P_2O_5 and K_2O(*a*)	Nitrogen treatment level (pounds per acre)	Derivative of polynomial (bu. per acre)	Derivative of exponential (bu. per acre)
1	20	.076	.049
1	40	.067	.032
1	80	.050	.023
1	120	.032	.021
1	160	.014	.019
1	200	−.003	.019
1	240	−.021	.018
2	20	.074	.051
2	40	.065	.033
2	80	.047	.024
2	120	.030	.021
2	160	.012	.020
2	200	−.006	.019
2	240	−.023	.019
3	20	.067	.053
3	40	.058	.035
3	80	.041	.025
3	120	.023	.022
3	160	.005	.021
3	200	−.012	.020
3	240	−.030	.020

(*a*) Nitrogen is varied with P_2O_5 and K_2O fixed at three levels: (1) 40-20, (2) 160-80, and (3) 480-240 pounds per acre respectively.

of the two functions produce different estimates of the productivity of the various plant nutrients. For example, the derivative of the polynomial indicates that the marginal productivity of nitrogen over the range of 30 to 100 pounds, which is a common range of application, is almost double the marginal productivity schedule generated by the derivative of the exponential.

Derivatives of the two functions with respect to P_2O_5 also exhibit substantial differences over the range of usual applications. However, the marginal productivity of phosphorus is low and the absolute value of the differences between the two derivatives is small as is shown in Table 8 and Fig. 3. Coefficients for K_2O variables in both equations lack statistical significance at any acceptable probability level.

Maximum Yields and High-Profit Plant Nutrient Applications

Maximum wheat yields of about 39 bushels per acre were predicted using the polynomial equation. This yield occurs with plant nutrient applications of about 195 pounds of N, 300 pounds of P_2O_5, and 60 pounds of K_2O. The maximum yield predicted using the exponential is in excess of any yield observed in the experiment. It requires plant nutrient applications in excess of any quantity used in the experiment.

TABLE 9—Changes in wheat yields resulting from unit changes in P_2O_5 applications(a)

Treatment level of N and K_2O	P_2O_5 Treatment level (pounds per acre)	Derivative of polynomial (bu. per acre)	Derivative of exponential (bu. per acre)
1	40	.014	.012
1	80	.011	.006
1	160	.006	.002
1	240	.000	.001
1	320	−.005	.001
1	400	−.011	.000
1	480	−.017	.000
2	40	.016	.013
2	80	.013	.006
2	160	.008	.003
2	240	.002	.001
2	320	−.003	.001
2	400	−.009	.000
2	480	−.015	.000
3	40	.022	.015
3	80	.019	.007
3	160	.013	.003
3	240	.008	.002
3	320	.002	.001
3	400	−.004	.000
3	480	−.009	.000

(a) P_2O_5 is varied with N and K_2O fixed at three levels: (1) 20-20, (2) 80-80, and (3) 240-240 pounds per acre respectively.

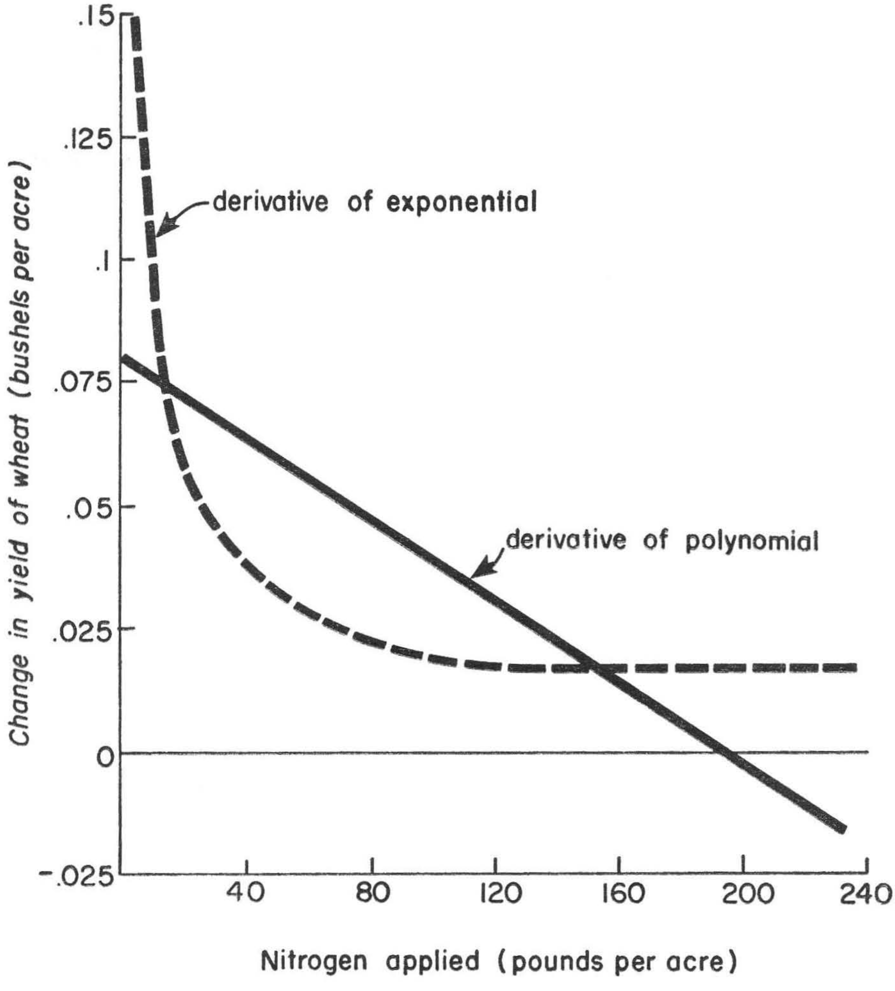

Fig. 2. Partial derivatives of the polynominal and exponential functions for wheat with respect to nitrogen.

As the predicted maximum yield and the plant nutrient input producing this yield lie beyond the range of observed values, no valid inferences can be made about these predictions.

Both functions generated response surfaces which illustrated substantial positive yield response to N. However, as the response surfaces had only moderate slopes, relatively small applications of nitrogen were profitable. The predicted high-profit nitrogen applications with varying nitrogen and wheat prices are shown in Table 10. As with oats, larger nitrogen applications are indicated as optimum by the polynomial function than by the exponential function.

Analysis of the Corn Data

Two corn crops have been produced and harvested in the rotation experiment. The corn plots were located at the Calhoun county site in 1955.[19] A severe summer drought reduced yields of corn in this

[19]Michigan 250 was the variety of corn produced on the rotation experiment plots in both 1955 and 1956.

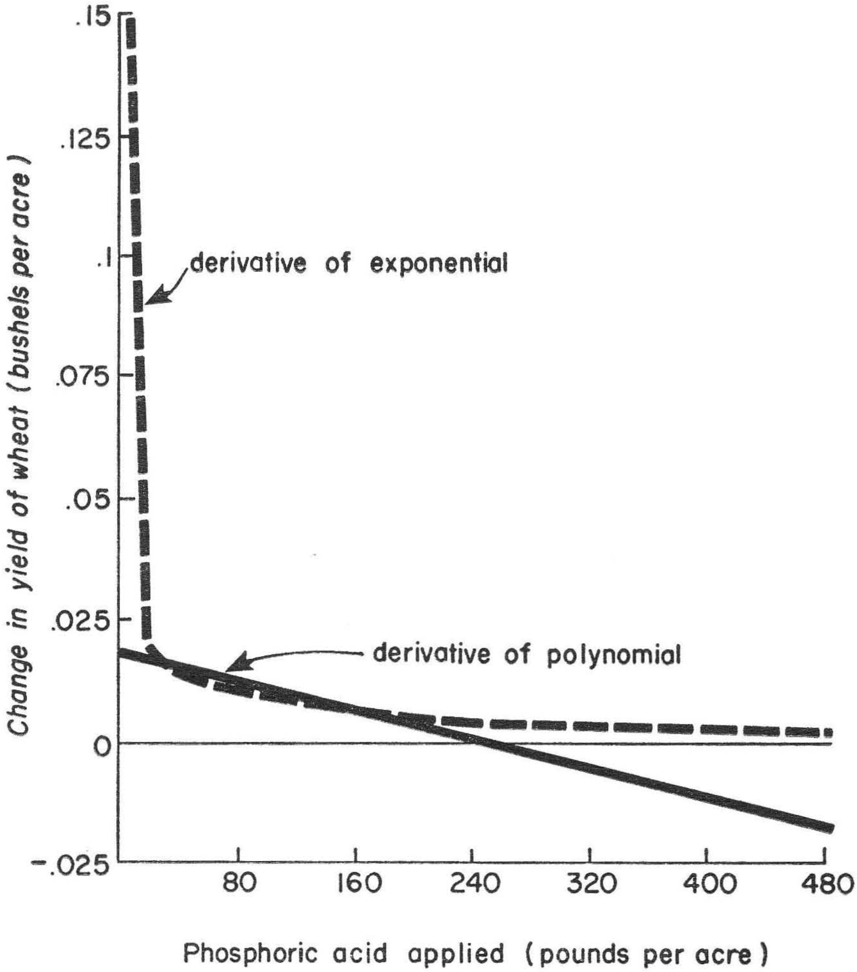

Fig. 3. Partial derivatives of the polynominal and exponential functions for wheat with respect to phosphoric acid.

TABLE 10—Predicted optimum nitrogen applications(a) with varying nitrogen and wheat prices

Price of wheat (per bushel)	Price of nitrogen							
	$0.09		$0.11		$0.13		$0.15	
	Poly.	Exp.	Poly.	Exp.	Poly.	Exp.	Poly.	Exp.
$1.50	24	13	8	12	0	9	0	8
$1.75	35	17	21	13	7	11	0	9
$2.00	43	20	31	15	19	12	7	11
$2.25	49	22	38	18	28	14	17	12
$2.50	54	23	45	20	35	16	25	13

(a) These optimum nitrogen applications were computed with no applications of P_2O_5 and K_2O. If P_2O_5 and K_2O applications were fixed at 160 and 80 pounds respectively, optimum nitrogen application rates would have been from 2 to 5 pounds smaller. These application rates were computed using 1956 experimental data only and are not to be interpreted as being recommended treatment rates.

area particularly on the lighter upland soils. An extensive analysis of the 1955 corn data was conducted by Knetsch and others and was reported previously (4) and (5). Knetsch found that a Carter-Halter type exponential provided a better statistical fit to the data than did several other functions fitted. Significant response was found to exist only for applied nitrogen.

The fitted function is shown in Equation VII.

$$\text{Equation VII: } \hat{Y}_c = 37.71\,(N + 0.1)^{.18627}\,(.96230)^{(N+0.1)},$$

when N was measured in 20-pound units. The addition of 0.1 of a unit to nitrogen inputs alleviated the problem of forcing the function to have a value of zero when any one of the plant nutrient inputs was zero. The coefficient of multiple correlation for this equation was 0.69. The high-profit nitrogen application varied from 29 to 54 pounds per acre as the price of corn was varied from $0.80 to $2.00 per bushel with nitrogen priced at $0.15 per pound.

High-profit nitrogen applications varied from 42 to 67 pounds per acre with nitrogen prices at $0.09 per pound when the price of corn varied from $0.80 to $2.00 per bushel. Although high-profit nitrogen inputs were not large, these moderate applications resulted in rather substantial increases in net income per acre. A comparison of observed and predicted yields is shown in Table 11.

Corn was produced on the Kalamazoo county site in 1956. Once again the crop was damaged by a severe late summer drought. Check plot yields did not differ significantly from those receiving applied plant nutrients. Preliminary tabulations indicated very little association of yield variance with variance in any of the three applied nutrients. This lack of relationship was further substantiated by functional analysis. A nine-term polynomial was fitted to the data with the estimated parameters shown in Equation VIII.

$$\text{Equation VIII:}\quad \hat{Y}_c = 54.8040409 - \underset{(.03912675)}{.00457618\,N} + \underset{(.00014589)}{.00011376\,N^2}$$

$$\underset{(.01964316)}{.01418892\,P} + \underset{(.00003652)}{.00000931\,P^2} + \underset{(.03975044)}{.02691734\,K} - \underset{(.00146457)}{.00009253\,K^2} +$$

$$\underset{(.00004967)}{.00006697\,NP} + \underset{(.00009954)}{.00003572\,NK} + \underset{(.00004916)}{.00005602\,PK}$$

None of the parameters in this equation differ significantly from zero. This lack of significance is not surprising as the adjusted coefficient of multiple correlation for the equation is only 0.23 and the adjusted coefficient of multiple determination is only 0.05, a value that does not differ significantly from zero.

A Carter-Halter type equation which was fitted to the data is shown in Equation IX.

$$\text{Equation IX:}\quad \text{Log}\ \hat{Y}_c = 1.71995756 + \underset{(.01631768)}{.01275310\log N} - \underset{(.00012096)}{.00001894\,N}$$

$$+ \underset{(.01369492)}{.00578977\log P} - \underset{(.00005505)}{.00011081\,P} - \underset{(.01627420)}{.00531799\log K} + \underset{(.00011830)}{.00018587\,K}$$

Only the fourth term in this equation, P, is statistically significant. As the phosphoric acid variable is represented by two terms, one of

TABLE 11—Comparison of observed and predicted corn yields on a Kalamazoo sandy loam soil, 1955

Nitrogen per acre (pounds)	Number of plots	Average of actual yields (bu. per acre)	Predicted yield of corn (bu. per acre)	Marginal product of 20-pound units of nitrogen (bu. per acre)
0	18	26.3	25.8	0
20	24	40.6	38.2	12.4
40	14	43.5	41.8	3.6
80	29	43.4	44.1	1.15(*a*)
160	18	42.8	43.0	−0.50(*a*)
240	27	40.7	39.8	−0.85(*a*)

(*a*) Average marginal product of 20-pound units of nitrogen for the application intervals shown in column 1.

which is not significant, the total influence of phosphoric acid is of questionable statistical significance. None of the coefficients for nitrogen or potash differ significantly from zero.

About the same proportion of total variance in yield is associated with regression as it was for the polynomial. The adjusted coefficients of multiple correlation and multiple determination are 0.24 and 0.06 respectively. The only inference that appears to be warranted by these analyses is that no significant part of the variance in corn yields was associated with applied plant nutrients.

In summary, only moderate applications of nitrogen, 30 to 65 pounds per acre at extreme nitrogen-corn price ratios, were profitable on corn on Kalamazoo sandy loam in 1955. No plant nutrient applications were profitable in 1956 as a result of the weather that occurred in these 2 years.

Alfalfa

No alfalfa was grown in the first year of the experiment because of the inability to establish any harvestable growth in the first year of the experiment. In 1956, the stand of alfalfa was heterogeneous. Large adjoining areas in the field had moderately good stands while other areas had almost no alfalfa growing on them. No attempt was made to collect and analyze yield data because yield differences were obviously a function of differences in stand not associated with applied plant nutrients.

Analysis of the Continuous Corn Data

The initial corn crop in a continuous corn rotation was produced on a Wisner clay-load soil in Tuscola county in 1956.[20] Preliminary inspection of the data indicated small and heterogeneous yield responses to applied plant nutrients. The eight check plots in this experiment had an average yield of 100.6 bushels per acre, while the average of all 210 plots in the experiment was 109.7 bushels per acre.

A 9-variable polynomial was fitted to the data and the results of this formulation are shown in Equation X.

Equation X: $\hat{Y}_c = 104.56551027 + \underset{(.03470491)}{.06991434\,N} + \underset{(.01727950)}{.05075485\,P} - \underset{(.03485558)}{.00162951\,K} - \underset{(.00010828)}{.00035693\,N^2} - \underset{(.00002874)}{.00006895\,P^2} - \underset{(.00011605)}{.00005357\,K^2} - \underset{(.00003978)}{.00003969\,NP} + \underset{(.00007936)}{.00011205\,NK} + \underset{(.00004383)}{.00006075\,PK}$

[20]The variety of corn produced in this experiment in 1956 was Michigan 480.

Coefficients of four of the variables, N, N^2, P, and P^2, were significant at the 1 percent probability level, whereas none of the potash variables were statistically significant. Only a small portion of yield variance was associated with applied plant nutrients, as the coefficients of multiple correlation and multiple determination were only 0.40 and 0.16, respectively.

As none of the independent variables containing a potash term were statistically significant, the polynomial was reformulated, dropping the variables that contained a potash term. The shortened polynomial is shown in Equation XI.

$$\text{Equation XI} \quad \hat{Y}_c + 104.08269882 + \underset{(.03429868)}{.07370454\,N} + \underset{(.01711621)}{.05002273\,P} - \underset{(.00010726)}{.00033159\,N^2} - \underset{(.00002733)}{.00005602\,P^2} - \underset{(.00003896)}{.00002546\,NP}$$

In Equation XI, the first four coefficients are significant at the 1 percent probability level. The fifth term, a cross product, was not significant at any acceptable significance level. The coefficients of multiple correlation and multiple determination for the shortened polynomial were 0.39 and 0.16 respectively.

Because of the small portion of yield variance associated with applied plant nutrients (as indicated by inspection and the fitted polynomials) no attempt was made to fit an exponential type equation to the data.

Maximum Yield and High-Profit Combinations of Plant Nutrients

Coefficients for the nitrogen and phosphoric acid variables were similar for the two polynomials fitted to the data. As the potash coefficients were not significant, the plant nutrient combination providing maximum yields was restricted to N and P_2O_5 and was calculated from Equation XI. The maximum predicted yield, 123.4 bushels per acre, was obtained using 95 pounds of N and 425 pounds of P_2O_5. The cost of using any amount of applied plant nutrients exceeded the returns unless corn prices exceeded $1.60 per bushel and then only small applications of nitrogen were profitable.

The high check plot yields, in excess of 100 bushels per acre, indicate that the soil was quite fertile prior to additional applications of plant nutrients, although soil tests indicated only a moderate fertility level. Other possible sources of yield variance were present in the experimental field, including differences in previous cropping history.

Although yields from the plot areas with different cropping histories were not statistically different, this factor of heterogeneity may have contributed some variance to crop yields.

Analysis of the Bean Data from the Corn, Bean, and Wheat Rotation

Field beans[21] were produced on a Sims loam soil in Gratiot county in 1956. The bean crop is part of an intensive cash crop rotation of corn, beans, and wheat. Experimental plots had received plant nutrient treatments in 1955 identical to the 1956 treatments. Thus, some residual fertility might have been expected to be present in 1956, particularly on plots that received heavy applications of fertilizer the previous year. Preliminary tabulation of the data indicated a substantial response to nitrogen applications, a smaller response to phosphoric acid, and no appreciable response to applied potash.

Three functions were fitted to the bean data. The first two are exponential type formulations and the third a 5-variable polynomial. The original production function formulation is a 6-variable exponential of the Carter-Halter type. Although preliminary analysis had indicated no response to potash, variables containing potash terms were included in this original exponential shown in Equation XII.

Equation XII: $$\text{Log } \hat{Y}_b = 1.2034797 + \underset{(.01752903)}{.03281226} \log N + \underset{(.0010894)}{.00039897} N + \underset{(.01558938)}{.01952743} \log P + \underset{(.00004950)}{.00006227} P + \underset{(.01859111)}{.00188061} \log K + \underset{(.00006852)}{.00005091} K$$

The adjusted coefficient of multiple correlation for this equation was 0.61 and the coefficient of multiple determination was 0.37. This indicates that about 37 percent of the variance in bean yields was associated with regression. Because of the large standard errors for the potash coefficients a second formulation of the exponential was made with the potash terms dropped. This exponential is shown in Equation XIII.

Equation XIII: $$\text{Log } \hat{Y}_b = 1.20741357 + \underset{(.01667665)}{.03473935} \log N + \underset{(.00010650)}{.00039659} N + \underset{(.01460961)}{.02146077} \log P + \underset{(.00004835)}{.00005973} P$$

The adjusted coefficient of multiple correlation for the shortened exponential was again 0.61 and the coefficient of multiple determina-

[21]Sanilac beans were produced in this experiment.

tion was 0.37. Coefficients of the nitrogen and phosphoric acid variables were not changed appreciably by omitting the nonsignificant potash terms. Phosphoric acid terms were not significant at the 10 percent probability level. However, the magnitude of the estimated coefficients for these terms exceeded their respective standard errors. Finally, a 5-variable polynomial was fitted to the bean data. The results are shown in Equation XIV.

Equation XIV: $\hat{Y}_b = 17.6023144 + \underset{(.0114222)}{.06268789}\ N - \underset{(.00003541)}{.00010708}\ N^2 + \underset{(.00580265)}{.01274996}\ P - \underset{(.00000873)}{.00001056}\ P^2 + \underset{(.00001309)}{.00000634}\ NP$

The adjusted coefficients of multiple correlation and determination for this equation were 0.65 and 0.42, respectively.

A comparison of observed and predicted yields using the three functions fitted to the data are presented in Table 12. As in previous instances, inspection of the residual quantities (i.e., differences between predicted and observed values) of the three functions provides little basis for choosing any one function over the others. This is true because of the relative uniformity of the magnitude and direction of the residuals. Partial derivatives of the three functions with respect to nitrogen are shown in Table 13 and Fig. 4. Partial derivatives with respect to phosphoric acid are presented in Table 14 and Fig. 5.

Maximum Yields and Optimum Inputs of Plant Nutrients

Derivatives of the two exponential equations with respect to nitrogen are characterized by properties that are unusual for marginal product schedules. These derivatives (shown in Table 13 and Fig. 4) first exhibit a range of diminishing values and then acquire values of increasing magnitude. Because the derivatives of the exponential exhibit this rather illogical property of diminishing returns followed by increasing returns to successive nitrogen inputs, the polynomial equation is probably a more appropriate approximation of the fertilizer response surface.

The maximum predicted yield as calculated from the polynomial equation is 32.2 bushels per acre. This maximum is achieved by using slightly less than 318 pounds of nitrogen and about 629 pounds of P_2O_5. The quantities of N and P_2O_5 that produce the maximum bean yield are almost identical with those of the highest treatment level in the experiment.

TABLE 12—Observed and estimated bean yields, 1956

Treatment (pounds per acre)			Predicted yield(*a*)			Observed yield(*b*)	Residual(*c*) ($Yi-\hat{Y}i$)		
N	P_2O_5	K_2O	Poly.	Exp.(1)	Exp.(2)		Poly.	Exp.(1)	Exp.(2)
0	0	0	17.6	16.1	16.0	17.4	−0.2	1.3	1.4
20	40	20	19.3	19.8	19.6	25.4	6.1	5.6	5.8
20	40	160	19.3	19.8	20.0	25.9	6.6	6.1	5.9
20	40	320	19.3	19.8	20.4	19.6	0.3	−0.2	−0.8
20	320	20	21.9	21.5	21.2	15.4	−6.5	−6.1	−5.8
20	320	160	21.9	21.5	21.6	24.5	2.6	3.0	2.9
20	320	320	21.9	21.5	22.1	21.4	−0.5	−0.1	−0.7
20	640	20	22.7	22.9	22.5	25.8	3.1	2.9	3.3
20	640	160	22.7	22.9	23.0	21.0	−1.7	−1.9	−2.0
20	640	320	22.7	22.9	23.4	14.8	−7.9	−8.1	−8.6
40	80	40	21.0	21.1	20.9	21.9	0.9	0.8	1.0
80	160	80	23.9	23.0	22.8	31.1	7.2	8.1	8.3
160	40	20	25.6	24.2	23.8	23.8	−1.8	−0.4	0.0
160	40	160	25.6	24.2	24.3	26.9	1.3	2.7	2.6
160	40	320	25.6	24.2	24.8	27.8	2.2	3.6	3.0
160	320	20	28.4	26.3	25.8	31.6	3.2	5.3	5.8
160	320	160	28.4	26.3	26.4	33.8	5.4	7.5	7.4
160	320	320	28.4	26.3	26.9	25.6	−2.8	−0.7	−1.3
160	640	20	29.5	27.9	27.4	24.7	−4.8	−3.2	−2.7
160	640	160	29.5	27.9	28.0	33.5	4.0	5.6	5.5
160	640	320	29.5	27.9	28.5	29.3	−0.2	1.4	0.8
240	480	240	31.1	29.6	29.9	29.5	−1.6	−0.1	−0.4
320	40	20	27.6	28.7	28.2	32.4	4.8	3.7	4.2
320	40	160	27.6	28.7	28.8	27.5	−0.1	−1.2	−1.3
320	40	320	27.6	28.7	29.4	26.4	−1.2	−2.3	−3.0
320	320	20	30.7	31.2	30.6	34.1	3.4	2.9	3.5
320	320	160	30.7	31.2	31.2	29.7	−1.0	−1.5	−1.5
320	320	320	30.7	31.2	31.9	27.8	−2.9	−3.4	−4.1
320	640	20	32.2	33.1	32.5	34.8	2.6	1.7	2.3
320	640	160	32.2	33.1	33.1	33.7	1.5	0.6	0.6
320	640	320	32.2	33.1	33.8	30.6	−1.6	−2.5	−3.2

(*a*) Exp. (1) is the 4-term exponential and Exp. (2) is the 6-term exponential.
(*b*) The observed yield for the 0-0-0 treatment is an average of yields from 11 plots, all other observed yields are averages of 2 plots.
(*c*) Residuals are deviations of predicted yields from average observed yields.

The bean yield production surface generated by the polynomial equation is shown in Fig. 6. The slope of the response surface is quite steep in the nitrogen-yield dimension, but the slope of the surface in the phosphoric acid-yield dimension is relatively small.

Despite the large phosphoric acid inputs which produce maximum yields, applications of phosphoric acid were profitable only with extremely high bean prices and extremely low prices of P_2O_5. Even then, only small applications of P_2O_5 were indicated to be profitable. Nitrogen applications, on the other hand, were profitable over a wide range of bean and nitrogen prices. Predicted optimum applications of nitrogen ranged from a low of 62 pounds per acre with nitrogen priced at $0.15 cents per pound and beans at $4.00 per bushel to a high of 118 pounds per acre with nitrogen priced at $.09 cents per pound and beans worth $7.00 per bushel.

TABLE 13—Changes in bean yields resulting from unit changes in applied nitrogen

Treatment level of P_2O_5 and K_2O(*a*)	Nitrogen treatment level (pounds per acre)	Derivative of polynomial (bu. per acre)	Derivative of exponential (1) (bu. per acre)	Derivative of exponential (2) (bu. per acre)
1	20	.060	.052	.050
1	40	.055	.037	.035
1	80	.047	.030	.029
1	120	.038	.028	.027
1	160	.030	.027	.027
1	200	.021	.027	.027
1	240	.013	.027	.027
1	320	.005	.029	.029
2	20	.060	.055	.052
2	40	.056	.039	.037
2	80	.048	.031	.030
2	120	.039	.029	.029
2	160	.030	.029	.028
2	200	.022	.029	.028
2	240	.013	.029	.029
2	320	.004	.031	.031
3	20	.062	.057	.055
3	40	.058	.040	.039
3	80	.050	.032	.032
3	120	.041	.030	.030
3	160	.032	.030	.030
3	200	.024	.030	.030
3	240	.015	.030	.030
3	320	.002	.032	.032
4	20	.063	.060	.060
4	40	.059	.042	.042
4	80	.051	.034	.034
4	120	.042	.032	.033
4	160	.033	.031	.032
4	200	.025	.032	.032
4	240	.016	.032	.033
4	320	.001	.033	.035

(*a*) Nitrogen is varied with P_2O_5 and K_2O fixed at: (1) 40-20, (2) 160-80, (3) 320-160, and (4) 640-320, respectively. Derivatives of the polynomial and Exp. (1) are independent of applied K_2O as there were no K_2O variables in the functions for which these derivatives were taken.

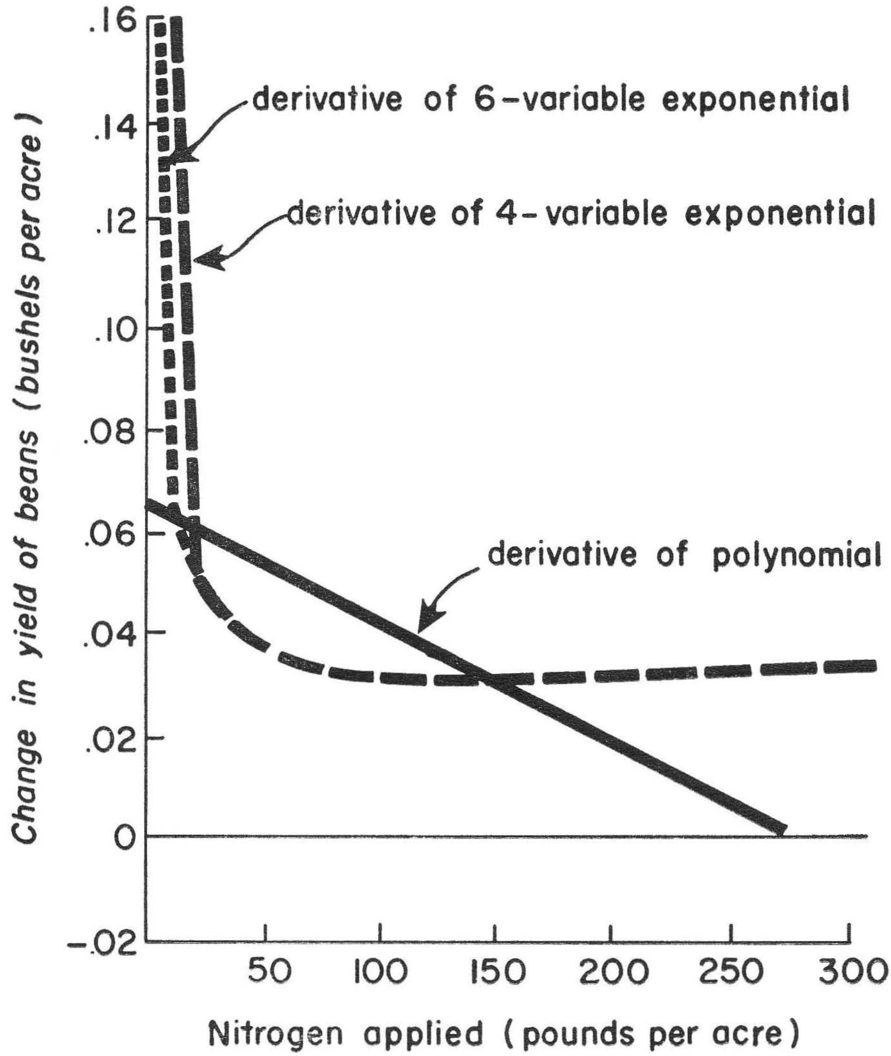

Fig. 4. Partial derivatives of a polynominal and two exponential functions for beans with respect to nitrogen.

The predicted optimum applications of plant nutrients over a wide range of fertilizer and bean prices are shown in Table 15. The estimated high-profit plant nutrient applications were all predicted from the polynomial equation.

EVALUATION OF PROCEDURES AND RESULTS

Evaluation of Experimental Designs

The experimental designs used in the several experiments described here were formulated with several restrictions and objectives in view. Prior to designing the experiments, it was decided that continuous function analysis of the experimental data would provide

TABLE 14—Changes in bean yields resulting from unit changes in applied phosphoric acid

Treatment level of N and K_2O(*a*)	P_2O_5 treatment level (pounds per acre)	Derivative of polynomial (bu. per acre)	Derivative of exponential (1) (bu. per acre)	Derivative of exponential (2) (bu. per acre)
1	40	.012	.013	.012
1	80	.011	.008	.008
1	160	.009	.006	.005
1	240	.008	.005	.005
1	320	.006	.004	.004
1	400	.004	.004	.004
1	480	.003	.004	.004
1	640	.001	.004	.004
2	40	.012	.015	.015
2	80	.012	.009	.009
2	160	.010	.006	.006
2	240	.008	.005	.005
2	320	.006	.005	.005
2	400	.005	.005	.005
2	480	.003	.005	.004
2	640	.000	.004	.004
3	40	.013	.016	.015
3	80	.013	.010	.010
3	160	.011	.007	.007
3	240	.009	.006	.006
3	320	.008	.005	.005
3	400	.006	.005	.005
3	480	.004	.005	.005
3	640	.000	.005	.005
4	40	.014	.019	.018
4	80	.013	.012	.012
4	160	.011	.008	.008
4	240	.010	.007	.007
4	320	.008	.006	.006
4	400	.006	.006	.006
4	480	.005	.006	.006
4	640	.001	.006	.006

(*a*) P_2O_5 is varied with N and K_2O fixed at (1) 20-20, (2) 80-80, (3) 160-160, and (4) 320-320 respectively. Derivatives of the polynomial and Exp. (1) are independent of applied K_2O as there were no K_2O variables in the functions for which these derivatives were taken.

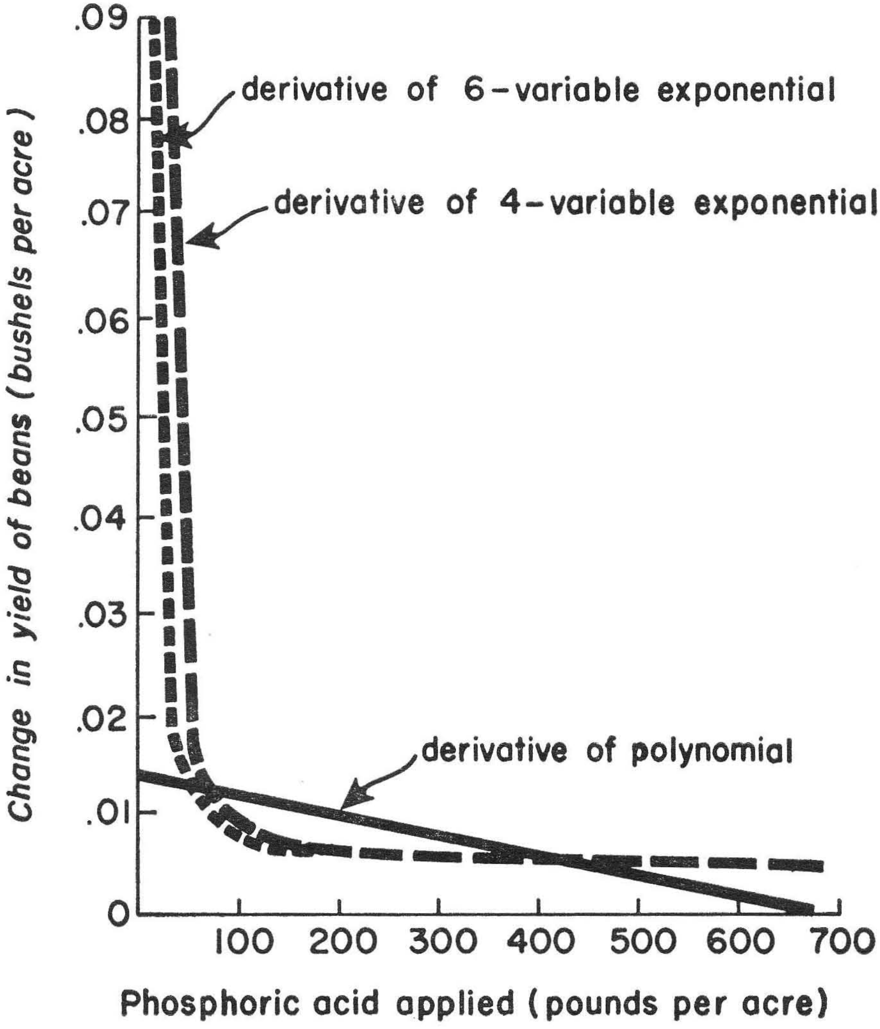

Fig. 5. Partial derivatives of a polynominal and two exponential functions for beans with respect to phosphoric acid.

a better basis for (1) estimating plant nutrient input-crop yield output coefficients and (2) facilitating an economic analysis to determine optimal plant nutrient applications, than would alternative methods of analysis. Thus the experiments were designed to provide data suitable for continuous function analysis.

Restrictions on funds, labor, and equipment limited the number and/or size of the experimental plots. Individual treatments or cells in the experimental designs were selected to: (1) describe the economically relevant portion of the production surface sufficiently to obtain reliable estimates of parameters of the production functions, (2) establish with adequate reliability the values for critical points on the production surfaces, e.g., origin of the functions and their inflection points, and (3) minimize intercorrelations among treatment variables.

It is the opinion of the authors that the experimental designs were quite satisfactory as a basis for providing data for continuous function analysis. The experimental designs utilized in the two original rotation experiments were not particularly efficient in providing data that readily facilitates estimation of (1) crop quality differences associated with treatments, (2) differences in plant nutrient content of plant tissue, and (3) differences in other plant and soil characteristics associated with plant nutrients but not associated in the manner postulated to exist for the basic input-output relationships.

Once committed to an incomplete factorial design with a minimum

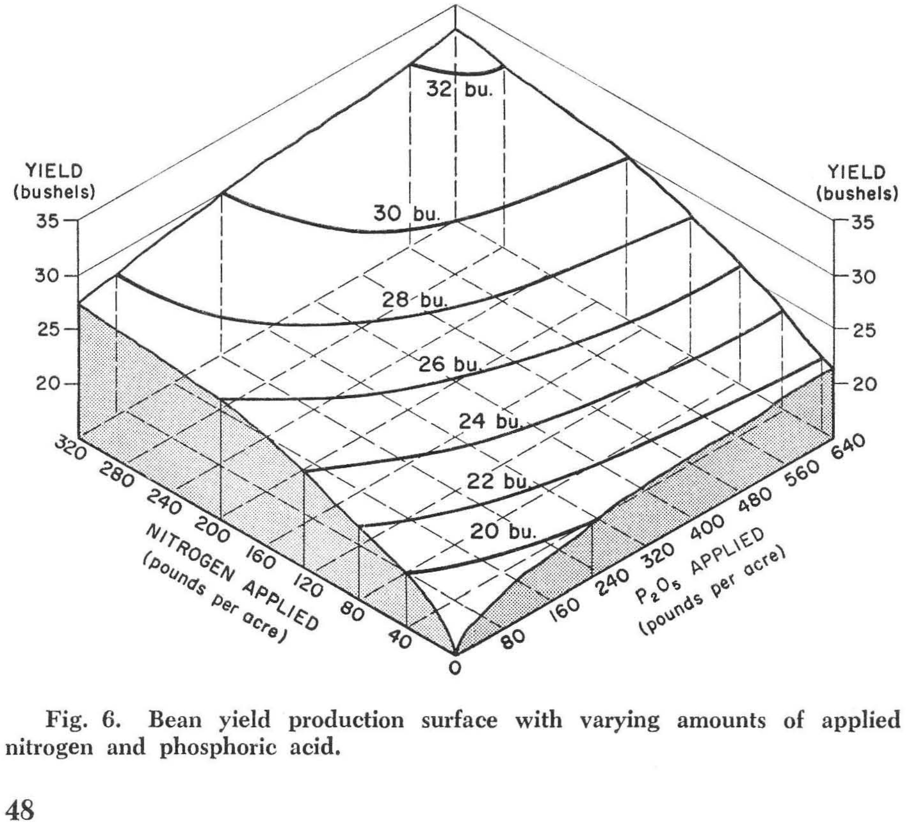

Fig. 6. Bean yield production surface with varying amounts of applied nitrogen and phosphoric acid.

TABLE 15—Predicted high profit fertilizer applications(a) for field beans with varying fertilizer and bean prices

Price of beans (per bushel)	Price of N (per pound)	Price of P_2O_5 (per pound)	Predicted high-profit applications of plant nutrients (pounds per acre)	
			N	P_2O_5
$4.00	$.15	$.11	62	0
	.13	.10	74	0
	.11	.09	85	0
	.09	.08	97	0
$4.50	.15	.11	72	0
	.13	.10	82	0
	.11	.09	92	0
	.09	.08	102	0
$5.00	.15	.11	80	0
	.13	.10	89	0
	.11	.09	98	0
	.09	.08	106	0
$5.50	.15	.11	86	0
	.13	.10	94	0
	.11	.09	102	0
	.09	.98	111	0
$6.00	.15	.11	91	0
	.13	.10	98	0
	.11	.09	106	0
	.09	.08	114	3
$6.50	.15	.11	95	0
	.13	.10	102	0
	.11	.09	109	0
	.09	.08	116	28
$7.00	.15	.11	99	0
	.13	.10	105	0
	.11	.09	112	14
	.09	.08	118	49

(a) These application rates were computed using 1956 experimental data only and are not to be interpreted as being recommended treatment rates.

number of replications, analysis of such factors as those listed above is quite difficult.[22] However, the designs that were used are adequate for these determinations if (1) the determinations can be made by correlation analysis or (2) if the determinations for one plant nutrient can be assumed to be independent of the treatment level of other

[22]The inference made here is that some of the determinations listed above can best be acquired by analysis of variance techniques.

plant nutrients. In the latter instance, this means that all observations for which the treatment levels of the studied variable are constant can be considered as replications of that treatment.

A modification of the incomplete factorial-minimum replication design used in the rotation experiments was incorporated into the continuous corn experiments as well as into experiments for potatoes and sugar beets. These designs include a triplicated factorial in addition to other treatments which were replicated twice. This modification was incorporated into the designs largely in order to facilitate analysis of by product data produced in the experiment. The experimental designs, as modified, still provide numerous nonreplicated treatments in order to specify the production surface adequately for continuous function analysis. Inclusion of a factorial in the experimental design facilitates utilization of analysis of variance techniques on a limited basis at little additional cost.

A possible criticism of the experimental designs that were used might be the large spacing between treatment levels of the various plant nutrients. Obviously, it would be desirable to have observations at treatment levels intermediate to those contained in the experiment. However, the experiments already were large and required considerable amounts of land, labor, machinery, equipment, and supervision.

Larger experiments would have created additional problems in conducting experimental work, such as seeding, harvesting, etc., with appropriate timeliness. The primary considerations in not enlarging the experiments by including intermediate treatment levels included additional time, land resources, and other costs necessary in such an expansion.

The correlation between applied and residual plant nutrients is relatively high in these experiments as individual plots receive the same treatment in successive years. A more comprehensive analysis of residual and applied plant nutrient relationships would be facilitated by rerandomizing treatments on the experimental fields.

Such a modification of the experimental design is currently being contemplated in order to provide observations over a much wider range in combinations of residual and applied plant nutrients.

Evaluation of Experimental Procedures

Mechanized procedures, to the extent feasible, were used in conducting the experimental work. When soil conditions allowed, plant nutrient applications were made with a 7-foot, tractor-drawn drill.

Seedings of small grains were also made with a 7-foot drill which required 1 round on the plots that were 14 feet wide. Wheat and oats were harvested with a 7-foot, self-propelled combine. Part of the corn crop was harvested by using an especially constructed single-row corn picker. When weather prevented applications of fertilizer and harvesting of corn by machine, the work was done by hand labor.

Some additional experimental error undoubtedly occurs because of use of machinery as compared with hand-labor methods. For example, plant nutrient applications are not weighed out precisely and delivered in exact amounts to individual plots. Small amounts of grain remain in the combine from one plot to another when harvesting and introduce some small experimental error. For the most part, however, these errors should average out and not bias the plant nutrient input-crop yield output estimates made.

Mechanization of experimental work provides some important implications, particularly with respect to the number and size of individual plots that can be included satisfactorily in an experiment. Two objectives of plant nutrient input-crop yield output research appear to be of relevance here. First, we want research results to be validly inferable to some farm population. Farmers typically operate as units fields of a minimum of several acres in size. The larger the experimental plots, the more nearly they tend to represent the conditions that exist on farms.

Farmers, and consequently researchers whose objective is to make input-output estimates applicable to farm conditions, are not particularly interested in measuring within treatment yield variance. Rather, they are interested in determining the variance in yield that can be attributed to variance in plant nutrient applications under farm conditions e.g., the change in yield resulting from application of an additional 20 pounds per acre of nitrogen, etc.

Researchers are interested, however, in having some assurance that within-treatment yield variance is not prohibitively large so as to constitute a large part of the total yield variance. Within-treatment variance tends to be reduced by increasing the size of individual experimental plots and the harvested portion of these plots provided soil variations do not increase appreciably as larger soil areas are incorporated into the experiment.

Increases in plot size are facilitated by mechanizing the experimental procedures used. Errors of inference caused by excessive within-treatment yield variance can be eliminated alternatively by replicating a given treatment several times and averaging the yields of the several replications. Additional replications of a treatment require more labor and have a higher cost than is true for a comparable enlargement of a given plot.

A second objective of input-output research, that of estimating productivity coefficients to which we can attach acceptable reliability measures, is aided by increasing the number of individual plots in an experiment. The standard error of estimated parameters in a functional equation diminishes as the number of observations increases. Attainment of both accurate and applicable research results, therefore, is enhanced by increasing the size and/or the number of experimental plots.

Evaluation of Analytical Procedures

The Continuous Function Analysis

A brief justification for utilizing continuous function analysis was presented earlier and will not be repeated here. Rather, a brief aposteriori evaluation of the effectiveness of the continuous function analysis used is attempted. Both polynomial and exponential type formulations of the respective production functions were fitted for all crops for which preliminary analysis indicated that an appreciable amount of variance in yield was associated with variance in applied plant nutrients.

No criteria are available to provide a basis for saying that one formulation is "absolutely" more appropriate than the other. However, some available measures do provide somewhat of a quantitative basis for comparison. Furthermore, logic and theory provide a basis for selecting one formulation in preference to the other in at least one instance.

Only a moderate amount of total yield variance was associated with applied plant nutrients. For the 9-variable polynomial for wheat when all individual observations were included, $\overline{R} = .66$ and $\overline{R}^2 = .44$ (Table 16). These values increased to .79 and .62, respectively, when the average yields from replicated plots were used as observations on the dependent variable in the analysis.

The effects on variance in crop yields of several factors such as experimental error, residual fertility, plant lodging, weed infestation, etc., are discussed elsewhere by Sundquist (1957) and are not treated here. As previously mentioned, comparison of the coefficients of multiple correlation for the two functions provides a guide as to the relative amount of yield variance associated with regression. This comparison is somewhat subjective, however, as:

(1) In the case of the exponentials, variance is measured in logarithms—but in the polynomials, it is measured in real numerical values. Although the logarithms and real numbers bear a consistent monotonic relationship to each other over the range of observed values, they do not retain a relationship of constant ratios.

(2) The two formulations differ as to the number of variables in the respective equations. Hence the number of degrees of freedom used in the two analyses differ slightly. The latter difficulty is not an important one, however, because of the large number of observations and, hence, of degrees of freedom, present in the analysis.

A comparison of the coefficients of multiple correlation and determination for the functions fitted is shown in Table 16. In two of the five comparisons a larger amount of yield variance is explained by regression for the exponential equations than for the polynomials. In one case, that of field beans, the polynomial equation has larger values of $\bar{R}$ and $\bar{R}^2$, whereas in the remaining two comparisons, values of

TABLE 16—Comparison of amounts of yield variance associated with alternative production function formulations

Crop	Function	Number of variables	$\bar{R}$	$\bar{R}^2$
Oats, 1956	Polynomial	9	.69	.48
	Exponential	6	.76	.58
Wheat, 1956	Polynomial	9	.66	.44
	Exponential	6	.65	.42
Corn, 1955	Exponential	6	.70	.47
	Polynomial(*a*)	9	.64	.41
Corn, 1956	Polynomial	9	.23	.05
	Exponential	6	.24	.06
Cont. Corn, 1956	Polynomial	9	.40	.16
	Polynomial	5	.39	.16
Beans, 1956	Polynomial	5	.65	.42
	Exponential	6	.61	.37
	Exponential	4	.61	.37

(*a*) The polynomial used on the 1955 Corn data was a square root polynomial of the form $Y = a + b_1N + b_2\sqrt{N} + b_3P + b_4\sqrt{P} + b_5K + b_6\sqrt{K} + b_7\sqrt{NP} + b_8\sqrt{NK} + b_9\sqrt{PK}$.

$\bar{R}$ and $\bar{R}^2$ for the two equations are almost identical. This comparison provides no very conclusive indication as to the superiority of either type of formulation.

A second comparison of the two types of functions was included in the analysis. Residual measures, $(Y_i - \hat{Y}_i)$, were computed for both types of functions. These residuals are measures of the deviation of predicted yields from observed yields. The residuals are almost identical for both types of functions for all crops. This is true for the magnitude of residuals as well as for their sign or direction. In summary, inspection and measurement of the residuals provides no discernable basis for choosing one function in preference to the other.

A third comparison of the polynomial and exponential functions that might provide some basis for choosing the more appropriate one is an inspection of the derivatives of these functions. Inspection of the partial derivatives of the exponential functions with respect to individual plant nutrients shows that the derivatives are usually of extreme magnitude (negative or positive) for small inputs of the plant nutrients and that they then become extremely small quite rapidly.[23]

Extremely large derivatives, $\frac{\partial Y}{\partial X_i}$, with small inputs of the X_i are partly a consequence of the yield being zero when any of the $X_i = O$. Derivatives of the polynomials, in comparison, usually take less extreme values.

It appears that over moderate plant nutrient input ranges for most crops, generally in the range of 20 to 150 pounds, the exponential is probably a satisfactory formulation of most of the input-output relationships. Derivatives of the exponentials for field beans, however, are contradictory to the usually accepted concept of diminishing returns. As a consequence, maximum yields predicted using the exponential functions were outside the range of observed inputs.[24]

Using the exponential type formulation in calculating the quantities of plant nutrients that result in maximum profits is a much more complex procedure than using a polynomial. Solving the exponential for optimal inputs requires use of a series of successive approximations known as Newton's method (Halter et al., 1957). This method re-

[23]There are exceptions to this statement. For example, the partial derivatives of bean yields with respect to nitrogen decrease at first and then increase with additional nitrogen inputs. There are other exceptions to this statement as well.

[24]The phenomena of maximum predicted yields being outside of the range of observed inputs is a criticism of the function only if in reality the maximum yield does occur within the range of observed inputs and is fallaciously predicted to be outside. If, indeed, the true maximum yield exists beyond the range of observed inputs, it is the experimental design, not the function, that should be criticized.

quires in part a graphic approximation refined by solving a series of equations. Statistical estimates of the parameters of both types of functions are rather easily acquired by methods of least squares.

The primary advantage of the exponential type formulation as compared with the particular polynomial used appears to be that it permits derivatives, $\frac{\partial Y}{\partial X_i}$, to take on nonlinear forms. Derivatives of a polynomial containing only first- and second-degree terms necessarily are restricted to linear form. It is the opinion of the authors that until easier computing procedures for solving a Carter-Halter type exponential for optimal plant nutrient inputs are available, modifications of the polynomial type formulation might be more desirable. Incorporating variables of degree $\neq$ to 1 or 2 results in nonlinear derivatives.

Economic Interpretation and Evaluation of Results

The most profitable amounts of plant nutrients to apply in the years for which data were available were computed for all crops except alfalfa. The analysis presented earlier indicated a significant response to nitrogen for corn produced on a Kalamazoo sandy loam soil in 1955. Significant yield responses to applied nitrogen were recorded for oats, wheat, and field beans, and for corn produced on a Wisner clay loam soil in 1956. The only crop that did not show a significant response to nitrogen in 1956 was corn produced on a Kalamazoo sandy loam soil.

Statistically significant response to applied phosphoric acid was recorded for field beans and corn produced on a Wisner clay loam soil in 1956. None of the crops showed significant yield response to applied potash. This lack of response to potash is not uncommon particularly for specific individual years.

Despite the several significant responses recorded, only moderate applications of plant nutrients were indicated to be profitable.[25] Small to moderate applications of nitrogen were profitable for five of the six crops produced if crop prices were sufficiently high and nitrogen prices sufficiently low. Applications of phosphoric acid were profitable only for beans in 1956 and then only with extremely high bean prices and extremely low prices for phosphoric acid.

[25]No credit was given for residual fertility values or for the benefits derived by grass or legume seedings because these values could not be readily ascertained.

The computed optimum applications shown in this publication are from limited experimental data and are not to be interpreted as recommended applications for several reasons. First, the 1955 and 1956 growing seasons were characterized by severe summer droughts. Thus the responses recorded may not typify the long-run expected responses to applied plant nutrients. Additional data collected over time are needed to obtain a probability distribution of yield responses over the range of existing weather conditions.

As a further qualification, it should be noted that the experimental results reported in the preceding analysis were obtained from soils that either were (1) relatively unproductive, as in the case of the Kalamazoo sandy loam soil or (2) relatively heavy and productive as the Sims loam and Wisner clay loam soils. One might expect, *a priori,* to obtain the greatest yield response to applied plant nutrients from soils with high productive potentials but low fertility levels. Greater yield response may be noted in future years on plots with a low nutrient level as residual fertility is depleted.

General Considerations

The analysis of experimental work presented here is rather limited in scope with respect to numbers of soils, crops, and growing seasons. Additional work is needed before the optimum plant nutrient treatments estimated here can be substantiated or invalidated as long-run optimum applications. Thus the limited experimental results presented here might be considered as illustrative of the analytical techniques used rather than indicative of responses typical for the crops considered. The distribution of yield responses over time is likely to be characterized by wide dispersions, particularly with lighter soils which are subject to frequent damage from drought.

A general implication posed by the experimental results is that, despite statistically significant yield responses, in several instances the cost of applying additional plant nutrients exceeded the value of the additional crop produced. This general result appears to indicate that analyses that only detect significant yield differences associated with plant nutrient applications are not adequate procedures for determining the most profitable rates of application. This result in itself would appear to substantiate the need for acquiring data to which economizing principles may be applied.

Finally, at the farm-management-application level of fertilization practices, these practices cannot be considered independent of other alternative farm business expenditures nor can they be considered independent of the many factors with which they interact. For example, a livestock farmer may find it profitable to fertilize oats, not for the oat yield benefits but in order to establish the clover or grass seeding that is essential to his livestock enterprise. However, if a farm manager is to intelligently and economically synthesize the costs and benefits of the many components of his farm business, he needs information as to the productivity of expenditures made for plant nutrients for the various crops he produces. Additional plant nutrient input-crop yield output estimates will help to provide this information.

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